MSRI2018, LECTURE 2

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2. Basic properties of $\mathcal{O}(\Omega)$

In this lecture we will study the standard local properties of holomorphic functions using the one variable results and calculus of several variables.

The space $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$ in the following sense. Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, each coordinate can be written as $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$. The mapping

$$z \mapsto (x_1, y_1, x_2, y_2, \dots, x_n, y_n) \in \mathbb{R}^{2n}$$

establishes an $\mathbb{R}$-linear isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, which is compatible with the metric structures.

The open ball of radius $r > 0$ centered at $a \in \mathbb{C}^n$ is defined by

$$B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}.$$

In several complex variables, it is often convenient to use another system of neighborhoods: the open polydiscs.

Definition 2.1 (Polydisc)**.** *An open polydisc centered at* $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ *and of polyradius* $r = (r_1, \dots, r_n)$, *where* $r_j \geq 0$ *for all* $0 \leq j \leq n$, *is given by the set*

$$P(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j \text{ for each } 0 \leq j \leq n\}.$$

It is quite obvious from the definition that an open polydisc is a Cartesian product of n planar open discs.

2.1. **Cauchy Integral Formula on Polydiscs.**

A lot of basic local properties of holomorphic function in one variable follows from the Cauchy Integral Formula which generalizes to polydiscs quite easily:

Theorem 2.2 (CIF on polydiscs)**.** *Let* $P = P(a, r)$ *be a polydisc in* $\mathbb{C}^n$ *and let* $f \in C(\overline{P}) \cap \mathcal{O}(P)$. *Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_0 P} \frac{f(\zeta)\, d\zeta_1 \dots d\zeta_n}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} \quad \text{for } z \in P, \tag{2.3}$$

where $b_0 P = \{\zeta \in \mathbb{C}^n : |\zeta_j - a_j| = r_j, 1 \leq j \leq n\}$.

Remark 2.4. It is important to note that the integral in not over the entire boundary of the polydisc. This part of the boundary $b_0 P$ is called the **distinguished boundary** of the polydisc.

Proof. For simplicity, we will prove this for $n = 2$. By Cauchy Integral Formula for one variable, if we fix z_2 such that $|z_2 - a_2| < r_2$, we have

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1 - a_1| = r_1} \frac{f(\zeta_1, z_2)}{(\zeta_1 - z_1)} d\zeta_1$$

for any z_1 with $|z_1 - a_1| < r_1$. Similarly for each fixed ζ_1,

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1 - a_1| = r_1} \int_{|\zeta_2 - a_2| = r_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 \, d\zeta_1$$

for each $(z_1, z_2) \in P$. The result for general n can be done using induction on n. □

ExerciseI: Show that if $z \in \overline{P}$ we have $|f(z)| \le |f|_{L^\infty(b_0 P)}$ for all $f \in \mathcal{O}(P) \cap C(\overline{P})$.

Theorem 2.5 (Cauchy estimates)**.** *Let $f \in \mathcal{O}(P(a, r))$. Then for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$,*

$$|D^\alpha f(a)| \le \frac{\alpha!}{r^\alpha} \|f\|_{L^\infty(P(a,r))}; \tag{2.6}$$

$$|D^\alpha f(a)| \le \frac{\alpha!(\alpha_1 + 2) \dots (\alpha_n + 2)}{(2\pi)^n r^{\alpha+2}} \|f\|_{L^1(P(a,r))}. \tag{2.7}$$

Here $D^\alpha = \dfrac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$, $\alpha! = \alpha_1! \dots \alpha_n!$, $r^\alpha = r_1^{\alpha_1} \dots r_n^{\alpha_n}$ and for $m \in \mathbb{Z}$, $\alpha + m = (\alpha_1 + m, \dots, \alpha_n + m)$.

Proof. Fix $0 < \rho < r$. Applying Theorem 2.2 to $P(a, \rho) \subset\subset P(a, r)$ and differentiating under the integral sign, we get

$$D^\alpha f(a) = \frac{\alpha!}{(2\pi i)^n} \int_{b_0 P(a,\rho)} \frac{f(\zeta) \; d\zeta_1 \dots d\zeta_n}{(\zeta - a)^{\alpha+1}}. \tag{2.8}$$

After an obvious estimatation of (2.8) and taking $\lim\limits_{\rho \to r}$ both sides we get (2.6). For (2.7), change to polar coordinates, multiply both sides of (2.8) by $\rho^{\alpha+1}$, and taking the estimate gives

$$|D^\alpha f(a)| \, \rho^{\alpha+1} \le \frac{\alpha!}{(2\pi)^n} \int\limits_{[0,2\pi]^n} \left| f(a + \rho e^{i\theta}) \right| \rho_1 \dots \rho_n d\theta_1 \dots d\theta_n. \tag{2.9}$$

Integrating (2.9) both sides over $0 \le \rho_j \le r_j$ for $1 \le j \le n$ gives (2.7). □

ExerciseI: Let $\{f_j\}_{j=1}^\infty \subset \mathcal{O}(\Omega)$ be a sequence of holomorphic functions converging compactly in Ω to f. Then prove that $f \in \mathcal{O}(\Omega)$, and for each $\alpha \in \mathbb{N}^n$, $\lim\limits_{j \to \infty} D^\alpha f_j = D^\alpha f$.

2.2. **Local power series expansion.**

Definition 2.10. *The multiple series $\sum_{\nu \in \mathbb{N}^n} b_\nu$ is called absolutely convergent if*

$$\sum_{\nu \in \mathbb{N}^n} |b_\nu| = \sup \left\{ \sum_{\nu \in \Lambda} |b_\nu| : \Lambda \text{ finite} \right\} < \infty.$$

A power series in n complex variables $z_1, \dots, z_n$ centered at $a \in \mathbb{C}^n$ is a multiple series $\sum_{\nu \in \mathbb{N}^n} b_\nu$ with

$$b_\nu = c_\nu (z - a)^\nu = c_{\nu_1 \dots \nu_n} (z_1 - a_1)^{\nu_1} \dots (z_n - a_n)^{\nu_n},$$

where $c_\nu \in \mathbb{C}$ for $\nu \in \mathbb{N}^n$. We will usually consider the power series centered at the origin.

Definition 2.11. *The domain of convergence Ω of the power series*

$$\sum_{\nu \in \mathbb{N}^n} c_\nu z^\nu \tag{2.12}$$

is the interior of the set of points $z \in \mathbb{C}^n$ for which (2.12) *converges absolutely.*

Lemma 2.13 (Abel's lemma). *Suppose $c_\nu \in \mathbb{C}$ for $\nu \in \mathbb{N}^n$ and that for some $w \in \mathbb{C}^n$*

$$\sup_{\nu \in \mathbb{N}^n} |c_\nu w^\nu| = M < \infty. \tag{2.14}$$

Let $r = \tau(w) = (|w_1|, \dots, |w_n|)$. Then the power series $\sum c_\nu z^\nu$ converges on the polydisc $P(0,r)$. Moreover, the convergence is ***normal*** *in the following sense: if $K \subset P(0,r)$ is compact and $\epsilon > 0$ is arbitrary, there is a finite set $\Lambda = \Lambda(K, \epsilon)$, such that*

$$\sum_{\nu \notin \Lambda} |c_\nu z^\nu| < \epsilon \quad \text{for all } z \in K.$$

Proof. To be given in Lecture 4. □

Theorem 2.15 (Taylor series). *Let $f \in \mathcal{O}(P(a,r))$. Then the taylor series of f converges to f absolutely and uniformly on compact subsets of $P(a,r)$, i.e.,*

$$f(z) = \sum_{\nu \in \mathbb{N}^n} \frac{D^\nu f(a)}{\nu!}(z-a)^\nu \quad \text{for } z \in P(a,r).$$

Proof. From the Cauchy integral formula (2.3), applied to $z \in P(a,\rho) \subset\subset P(a,r)$, we expand $(\zeta - z)^{-1} = (\zeta_1 - z_1)^{-1} \dots (\zeta_n - z_n)^{-1}$ into multiple geometric series

$$(\zeta - z)^{-1} = \sum_{\nu \in \mathbb{N}^n} \frac{(z-a)^\nu}{(\zeta - a)^{\nu+1}}, \tag{2.16}$$

which converges uniformly for $\zeta \in b_0 P(a,\rho)$, since $|z_j - a_j| / |\zeta_j - a_j| \le |z_j - a_j| / \rho_j < 1$ for such ζ and for all $1 \le j \le n$. Substituting (2.16) to (2.3) and interchanging summation and integration, we get

$$f(z) = \sum_{\nu \in \mathbb{N}^n} \left[\frac{1}{(2\pi i)^n} \int_{b_0 P(a,\rho)} \frac{f(\zeta) d\zeta_1 \dots d\zeta_n}{(\zeta - a)^{\nu+1}} \right] (z-a)^\nu \tag{2.17}$$

for $z \in P(a,\rho)$. By (2.8), the coefficient of $(z-a)^\nu$ in (2.17) equals $D^\nu f(a)/\nu!$. □

Theorem 2.18 (The identity theorem). *Let $\Omega \subset \mathbb{C}^n$ be a domain (open and connected) and let $f \in \mathcal{O}(\Omega)$. If $a \in \Omega$ is such that $D^\alpha f(a) = 0$ for all $\alpha \in \mathbb{N}^n$, then $f(z) = 0$ for $z \in \Omega$. In particular, if there is a nonempty open set $U \subset \Omega$, such that $f(z) = 0$ for $z \in U$, then $f \equiv 0$ on Ω.*

Proof. Theorem 2.15 implies that the set $D = \{z \in \Omega : D^\alpha f(z) = 0 \text{ for all } \alpha \in \mathbb{N}^n\}$ is open. By continuity of $D^\alpha f$, D is also closed. Since D is nonempty, connectedness implies $D = \Omega$. □

Theorem 2.19 (Open mapping theorem). *Let $\Omega \subset \mathbb{C}^n$ be domain and suppose $f \in \mathcal{O}(\Omega)$ is not constant. Then $f(U)$ is open for any open set $U \subset \Omega$.*

Proof. It suffices to show that for any open ball $B(a,r) \subset \Omega$, $f(B(a,r))$ is an open neighborhood of $f(a)$. Theorem 2.18 implies that $f|_{B(a,r)}$ is not constant, otherwise f would be constant on Ω. Choose $p \in B(a,r)$ such that $f(p) \neq f(a)$, and define $h(\lambda) = f(a + \lambda(p-a))$ for $\lambda \in \Delta = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. Then h is non constant on $\overline{\Delta}$ and holomorphic. By one variable open mapping theorem, $h(\overline{\Delta}) \subset f(B(a,r))$ is a neighborhood of $h(0) = f(a)$. □

Corollary 2.20 (The maximum modulus principle). *Let $\Omega \subset \mathbb{C}^n$ be an open set. Suppose $f \in \mathcal{O}(\Omega)$ and that $|f|$ has a local maximum at the point $a \in \Omega$. Then f is constant on the connected component of Ω containing a.*

Proof. ExerciseII.

□

3. Biholomorphic inequivalence of the ball and the polydisc

Definition 3.1 (Holomorphic map). *If* $\Omega \subset \mathbb{C}^n$ *is a domain, then* $F : \Omega \to \mathbb{C}^m$*, where* $F = (f_1, \dots, f_m)$*, is called a holomorphic map if each* f_j *is a holomorphic function for* $1 \le j \le m$*.*

The differential $dF(a)$ of a holomorphic map F at $a \in \Omega$ is a complex linear map $\mathbb{C}^n \to \mathbb{C}^m$, with the complex matrix representation

$$F'(a) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a) & \dots & \frac{\partial f_1}{\partial z_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial z_1}(a) & \dots & \frac{\partial f_m}{\partial z_n}(a) \end{bmatrix}$$

We call $F'(a)$ **the derivative (or complex Jacobian matrix) of the holomorphic map** F **at** a.

ExerciseII: Show that if $F : \Omega \to \mathbb{C}^n$ is a holomorphic map, where $\Omega \subset \mathbb{C}^n$ is a domain, then $\det J_{\mathbb{R}} F(z) = |\det F'(z)|^2 \ge 0$, where $J_{\mathbb{R}} F$ is the real $(2n \times 2n)$ Jacobian matrix of F.

Lemma 3.2 (Chain Rule). *Let* $D \subset \mathbb{C}^n$ *and* $\Omega \subset \mathbb{C}^m$ *be domains. If* $F = (f_1, \dots, f_m) : D \to \Omega$ *is holomorphic and* $g \in \mathcal{O}(\Omega)$*, then* $g \circ F \in \mathcal{O}(D)$*; moreover, for* $a \in D$ *and* $1 \le j \le n$*,*

$$\frac{\partial (g \circ F)}{\partial z_j}(a) = \sum_{k=1}^{m} \frac{\partial g}{\partial w_k}(F(a)) \frac{\partial f_k}{\partial z_j}(a). \tag{3.3}$$

Theorem 3.4 (Inverse Mapping Theorem). *Suppose* $\Omega \subset \mathbb{C}^n$ *is a domain and the holomorphic map* $F : \Omega \to \mathbb{C}^n$ *is non-singular at* a*, (i.e.,* $\det F'(a) \neq 0$*). Then there are open neighborhoods* U *of* a *and* W *of* $b = F(a)$*, such that* $F|_U : U \to W$ *is a homeomorphism with holomorphic inverse* $H : W \to U$*.*

If $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ are domains, then we say the map $F : \Omega_1 \to \Omega_2$ is **biholomorphic** if F is holomorphic homeomorphism with holomorphic inverse $F^{-1} : \Omega_2 \to \Omega_1$.

Two domains Ω_1 and Ω_2 are called **biholomorphically equivalent** or simply **biholomorphic** if there is a biholomorphism $F : \Omega_1 \to \Omega_2$.

The Riemann mapping theorem says that any simply connected domain in $\mathbb{C}$ (which is not whole of $\mathbb{C}$) is biholomorphic to the unit disc. H. Poincaré was the first to discover that the generalization fails in higher dimension:

Theorem 3.5. *There exists no biholomorphic map*

$$F : P(0,1) \to B(0,1)$$

between the polydisc and the unit ball in $\mathbb{C}^n$ *if* $n > 1$*.*

Proof. (Proof taken from [1])
For simplicity, we will assume $n = 2$. Let $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc in $\mathbb{C}$. Suppose $F = (f_1, f_2) : \Delta \times \Delta \to B = B(0,1) \subset \mathbb{C}^2$ be a biholomorphic map. For each fixed $w \in \Delta$, we define the holomorphic map $F_w : \Delta \to \mathbb{C}^2$ by

$$F_w(z) = \left(\frac{\partial f_1}{\partial w}(z, w), \frac{\partial f_2}{\partial w}(z, w) \right).$$

Let $\{z_\nu\} \subset \Delta$ be a sequence with $|z_\nu| \to 1$. We apply Montel's theorem to the bounded sequence of holomorphic maps $\{F(z_\nu, .)\}$ in the second variable to obtain a subsequence $\{z_{\nu_j}\}$, such that $\{F(z_{\nu_j}, .)\}$ converges compactly in Δ to a holomorphic map $\varphi : \Delta \to \overline{B}$. Since F is biholomorphic, we must have $F(z_{\nu_j}, w) \to bB$ for every $w \in \Delta$ as $z_\nu \to b\Delta$.

Hence $\varphi(\Delta) \subset bB$, i.e., if $\varphi = (\varphi_1, \varphi_2)$ then $|\varphi_1(w)|^2 + |\varphi_2(w)|^2 = 1$ for all $w \in \Delta$. Applying $\partial^2/\partial\overline{w}\partial w$ to this equation, we get $|\varphi_1'(w)|^2 + |\varphi_2'(w)|^2 = 0$, so $\varphi' \equiv 0$ on Δ. This gives

$$\lim_{z_\nu \to b\Delta} F_w(z) = 0.$$

This means F_w extends continuously to $\overline{\Delta}$, with boundary values 0. This is a contradiction because by maximum modulus principle, $F_w \equiv 0$ on Δ, which means $F(z, w)$ is independent of w, which means F cannot be injective. $\square$

References

[1] Range, R. Michael. **Holomorphic Functions and Integral Representation in Several Complex Variables**. Springer-Verlag.