MSRI2018, LECTURE 4

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1. Some basic definitions

We begin by stating a few definitions. A domain Ω in $\mathbb{C}^n$ is *Reinhardt* if whenever $z = (z_1, \ldots, z_n) \in \Omega$ for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $|\lambda_j| = 1$ for each j, we have

$$(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega.$$

A more sophisticated way of stating the definition of a Reinhardt domain is to state that it is invariant under the natural action of the n-dimensional torus group

$$\mathbb{T}^n = \{(\lambda_1, \ldots, \lambda_n) : |\lambda_j| = 1 \text{ for } j = 1, \ldots, n\}$$

which is an abelian group under multiplication.

Fundamental examples of Reinhardt domains in $\mathbb{C}^n$ are balls and polydiscs (centered at the origin), which have already been introduced in previous lectures. Recall that

$$\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$$

and

$$P(0,1) = \{z \in \mathbb{C}^n : \text{for } j = 1, \ldots, n \;\; |z_j| < 1\}.$$

are called the unit ball and polydisc in $\mathbb{C}^n$ respectively. We know from Lecture 2 that the ball and polydisc are not biholomorphically equivalent.

If Ω is a Reinhardt domain in $\mathbb{C}^n$, we denote by $|\Omega|$ the subset of $\mathbb{R}^n$ given by

$$|\Omega| = \{(|z_1|, \ldots, |z_n|) : (z_1, \ldots, z_n) \in \Omega\},$$

the *Reinhardt Shadow* of Ω. Notice that the Reinhardt Shadow $|\Omega|$ completely determines the set Ω. Therefore, Reinhardt domains in $\mathbb{C}^2$ and $\mathbb{C}^3$ can be visualized by looking at their shadows in $\mathbb{R}^2$ and $\mathbb{R}^3$. Notice also that a point $(r_1, \ldots, r_n)$ in the shadow $|\Omega|$ corresponds to the torus $\{(r_1 e^{i\theta_1}, \ldots, r_n r^{i\theta_n}) : \theta_1, \ldots, \theta_n \in \mathbb{R}\}$ in the domain Ω.

A Reinhardt domain Ω is said to be *complete Reinhardt* if $z = (z_1, \ldots, z_n) \in \Omega$ for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $|\lambda_j| \leq 1$ for each j, we have

$$(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega.$$

Notice that if a point $r = (r_1, \ldots, r_n) \in |\Omega|$, where Ω is complete Reinhardt domain, then the set

$$\overline{P(0,r)} = \{z \in \mathbb{C}^n : |z_1| \leq r_1, |z_2| \leq r_2, \ldots, |z_n| \leq r_n\}$$

is contained in Ω. We call $\overline{P(0,r)}$ the *closed polydisc* of polyradius $(r_1, \ldots, r_n)$. Its interior is the (open) polydisc of the same polyradius.

Again let Ω be a Reinhardt domain $\mathbb{C}^n$. Set

$$\Omega^* = \{(z_1, \ldots, z_n) \in \Omega : z_1 z_2 \ldots z_n \neq 0\},$$

and note that Ω^* is also Reinhardt and open. We denote by $\lambda(\Omega) \subset \mathbb{R}^n$ the set

$$\lambda(\Omega) = \{(\log|z_1|, \ldots, \log|z_n|) : (z_1, \ldots, z_n) \in \Omega^*\},$$

the *logarithmic shadow* of Ω. A complete Reinhardt domain is said to be *log-convex* or *logarithmically convex* if $\lambda(\Omega)$ is a convex subset of $\mathbb{R}^n$.

2. Power Series representation

We write a power series as

$$\sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha. \tag{2.1}$$

It is easy to define power series with other points as center.

Proposition 2.2. *Let $\Omega\subset\mathbb{C}^n$ be a complete Reinhardt domain, and let $f\in\mathcal{O}(\Omega)$. Then f admits a Taylor series representation which converges uniformly and absolutely on compact subsets of Ω.*

Proof. ExerciseII. □

Proposition 2.3 (Abel's lemma)**.** *Suppose $c_\nu\in\mathbb{C}$ for $\nu\in\mathbb{N}^n$ and that for some $w\in\mathbb{C}^n$*

$$\sup_{\nu\in\mathbb{N}^n}|c_\nu w^\nu| = M<\infty. \tag{2.4}$$

Let $r=\tau(w)=(|w_1|,\dots,|w_n|)$. Then the power series $\sum c_\nu z^\nu$ converges on the polydisc $P(0,r)$. Moreover, the convergence is ***normal*** *in the following sense: for $K\subset P(0,r)$ compact and some $\epsilon>0$, there is a finite set $\Lambda=\Lambda(K,\epsilon)$, such that*

$$\sum_{\nu\notin\Lambda}|c_\nu z^\nu|<\epsilon \quad \text{for all } z\in K.$$

Proof. Given $K\subset\subset P(0,r)$, choose $0<\lambda<1$, such that $K\subset P(0,\lambda r)$. For $z\in P(0,\lambda r)$ one obtains from (2.4) that

$$|c_\nu z^\nu|\le |c_\nu w^\nu|\,\lambda^{|\nu|}\le M\lambda^{|\nu|}\quad \text{for } \nu\in\mathbb{N}^n.$$

Since $\sum_{\nu\in\mathbb{N}^n}\lambda^{|\nu|}=(\sum_{j=0}^\infty\lambda^j)^n<\infty$, the result follows.

□

Definition 2.5. *The* domain of convergence *of the power series* (2.1) *is the open set which is the interior of the set of points $z\in\mathbb{C}^n$ where* (2.1) *converges* absolutely.

It is clear that the domain of convergence of a power series is a complete Reinhardt domain.

Theorem 2.6 (F. Hartogs, 1906)**.** *A bounded complete Reinhardt domain is the domain of convergence of a power series if and only if it is logarithmically convex.*

Remark. In fact the theorem holds also for unbounded complete Reinhardt domains. For simplicity, we consider the bounded case.

Proof. First suppose that we are given the power series (2.1), and we want to show that its domain of convergence Ω (which is a complete Reinhardt domain) is actually log-convex. Let $z,w\in\Omega$ and let $t\in[0,1]$. Then by applying Hölder's inequality with $p=\frac{1}{t}$ and $q=\frac{1}{1-t}$, we obtain that

$$\sum_\alpha |a_\alpha|\,|z^\alpha|^t\,|w^\alpha|^{1-t}=\sum_\alpha|a_\alpha z^\alpha|^t\,|a_\alpha w^\alpha|^{1-t}\le\left(\sum_\alpha|a_\alpha z^\alpha|\right)\left(\sum_\alpha|a_\alpha w^\alpha|\right)<\infty.$$

It follows that the point $t\lambda(z)+(1-t)\lambda(w)\in\Omega$, i.e., $\lambda(\Omega)$ is convex.

Now let Ω be a bounded log-convex Reinhardt domain. We want to show that there is a function $f\in\mathcal{O}(\Omega)$ whose Taylor expansion converges absolutely precisely on Ω. To construct this function, let for $\alpha\in\mathbb{N}^n$,

$$c_\alpha=\sup_{z\in\Omega}|z^\alpha|.$$

This is finite since Ω is bounded. Now consider the series

$$\sum_{\alpha\in\mathbb{N}^n}\frac{1}{c_\alpha}\cdot z^\alpha.$$

Since each term of this series is bounded by 1 on Ω, it follows from Abel's lemma that the series is absolutely convergent at each point on Ω, i.e., Ω is contained in the domain of convergence. Now we want to show that no point w outside $\overline{\Omega}$ belongs to the domain of convergence. Since the domain of convergence is open and Reinhardt, it is no loss of generality to assume that each coordinate $w_j > 0$. Then the point $(\log w_1, \dots, \log w_n)$ is a point of $\mathbb{R}^n$ outside the closure of the convex set $\lambda(\Omega)$, and therefore, can be separated from $\lambda(\Omega)$ by a hyperplane. Consequently, there is a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that

$$\ell(\lambda(w)) = \ell(\log w_1, \dots, \log w_n) > \sup_{\zeta\in\lambda(\Omega)} \ell(\zeta).$$

Let us write $\ell(u_1, \dots, u_n) = \beta_1 u_1 + \cdots + \beta_n u_n$. We claim that without loss of generality we can take the β_j's to be positive integers. First note that since Ω is complete Reinhardt, the set $\lambda(\Omega)$ contains an "infinite cube" of the form $(-\infty, c)^n$ for some $c < 0$. It follws that none of the numbers β_j can be negative, since, then by taking the corresponding u_j large and negative, we can make $\ell(u) > \ell(\lambda(w))$. Further, it is clearly possible to perturb the separating hyperplane slightly, to ensure that the β_j's are rational. Finally, multiplying by the common denominator, we may assume that each β_j is a positive integer. Now exponentiating, we see that $w^\beta > c_\beta$, and therefore, for each positive integer k, if $k\beta = (k\beta_1, \dots, k\beta_n)$, we have

$$w^{(k\beta)} = (w^\beta)^k > (c_\beta)^k = \left(\sup_{z\in\Omega}\left|z^\beta\right|\right)^k = c_{k\beta}.$$

Consequently, infinitely many terms of the series $\sum \frac{w^\alpha}{c_\alpha}$ exceed 1 in absolute value, and the series diverges at the point w.

□

Proposition 2.7. *Let $\Omega \subset \mathbb{C}^n$ be a complete Reinhardt domain. Then each holomorphic function on Ω extends holomorphically to the log-convex envelope of Ω.*

Therefore, if log-convex envelope of Ω is strictly larger than Ω, we have another example of Hartogs phenomenon.

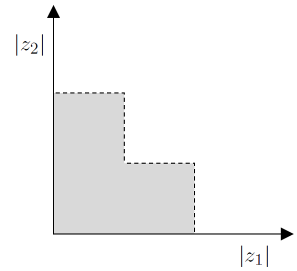
Proof. Let $f \in \mathcal{O}(\Omega)$. Then each $f \in \mathcal{O}(\Omega)$ has a power series representation, which converges in a log-convex domain containing Ω, and therefore on the log-convex envelope of Ω. The sum of the power series thus defines the required extension. □

Example: Consider the domain in $\mathbb{C}^2$ given by

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < \frac{1}{2} \right\},$$
$$\Omega_2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1, |z_1| < \frac{1}{2} \right\}.$$



Find the log-convex envelope of Ω.

3. Domains of holomorphy

Definition 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $f \in \mathcal{O}(\Omega)$. If $p \in b\Omega$, we say that f* extends holomorphically to p, *if there is a neighborhood U of p in $\mathbb{C}^n$, and a holomorphic $\widetilde{f} \in \mathcal{O}(U)$ which agrees with f on a component V of $U \cap \Omega$, where p lies in the closure of V.*

Definition 3.2. *A domain $\Omega \subset \mathbb{C}^n$ is said to be a* weak domain of holomorphy *if for each point $p \in b\Omega$, there is a function $f_p \in \mathcal{O}(\Omega)$ which does not extend to p holomorphically.*

Ω is said to be a domain of holomorphy *if there is a single function $f \in \mathcal{O}(\Omega)$ which has "natural boundary" $b\Omega$, i.e. for each point $p \in b\Omega$, the function f does not extend holomorphically to p.*

We will show later (lecture 12) that each weak domain of holomorphy is automatically a domain of holomorphy.

Exercise: (Runge) Show that each domain in the plane is a domain of holomorphy.

The *Levi Problem* asks for a geometric characterization of domains of holomorphy. Of course this is interesting only if the dimension $n \geq 2$. The solution is in terms of the geometric property known as "pseudoconvexity".

Proposition 3.3. *Every convex domain in $\mathbb{C}^n$ is a weak domain of holomorphy.*

Proof. Since Ω is convex, by a well-known geometric property of convex sets in real vector spaces, each point $p \in b\Omega$ admits a *supporting hyperplane*, i.e., there is an $\mathbb{R}$-linear functional $\ell : \mathbb{C}^n \to \mathbb{R}$ such that for all $z \in \Omega$, we have $\ell(z) < \ell(p)$, so that Ω lies entirely on one side of the hyperplane $\{z \in \mathbb{C}^n | \ell(z) = \ell(p)\}$. Write $\ell(z) = \sum_{j=1}^n \alpha_j z_j + \sum_{j=1}^n \beta_j \overline{z_j}$, and note that since ℓ is real-valued, we must have $\beta_j = \overline{\alpha_j}$ for each j, so $\ell(z) = \mathrm{Re}(h(z))$, where the complex linear functional $h : \mathbb{C}^n \to \mathbb{C}$ is given by $h(z) = 2\sum_{j=1}^n \alpha_j z_j$. Now let $f_p = (h - h(p))^{-1}$, which is a holomorphic function on Ω which cannot be extended to p even locally. □

Exercise: Show that the domain $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{\frac{1}{4}} < 1\}$ is a weak domain of holomorphy.

Exercise: Give an example of a non-convex domain of holomorphy in $\mathbb{C}^2$.

The following is a classical theorem.

Theorem 3.4 (H. Cartan and P. Thullen, 1932)**.** *A complete Reinhardt domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is log-convex.*

One part of this is clear: suppose that Ω is a complete Reinhardt domain which is a domain of holomorphy. Then there is a function $f \in \mathcal{O}(\Omega)$ with natural boundary $b\Omega$. The power series of f converges on Ω, but not on any larger domain, so Ω must be the domain of convergence of the power series of f. It follows that Ω is log-convex.

To prove the converse, we need to construct on each log-convex complete Reinhardt domain Ω a function with natural boundary on $b\Omega$. Cartan and Thullen did this by a several variable version of the argument for one variable. However, we will follow a different route:

(1) We will show that a log-convex Reinhardt domain is pseudoconvex.

(2) We will then show that on a bounded pseudoconvex domain in $\mathbb{C}^n$, the $\bar{\partial}$-problem can be solved.

(3) It will follow that for any pseudoconvex domain in $\mathbb{C}^n$, the $\bar{\partial}$-problem can be solved in the category of smooth functions.

(4) We will use this to show that each pseudoconvex domain is a weak domain of holomorphy.

(5) Finally, we will show that each weak domain of holomorphy is a domain of holomorphy, by using again the classical argument of Cartan and Thullen.