# MSRI2018, LECTURE 3

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**READ WITH CAUTION, TYPOS ABOUND.**

## 3. Plurisubharmonic functions

3.1. **Cauchy-Riemann complex III.** A given $(p,q)$-form has many equivalent representations of the form (1.4) of Lecture 1, but anti-commutivity of the wedge product produces a unique expression by summing only over *increasing* multi-indices. Write

$$\beta = \sum_{\substack{|I|=p\\|J|=q}}{}' \tilde{\beta}_{IJ}\, dz^I \wedge d\bar{z}^J \tag{3.1}$$

where $\sum'$ denotes summation over increasing multi-indices of the noted length, i.e., over $I=(i_1,\dots,i_p)$ and $J=(j_1,\dots j_q)$ where

$$1 \le i_1 < i_2 < \cdots < i_p \le n \quad \& \quad 1 \le j_1 < j_2 < \cdots < j_q \le n.$$

3.1.1. *$\bar\partial$ on $(p,q)$-forms.* The $\bar\partial$ operator defined on functions (now also called $(0,0)$-forms) extends to higher $(p,q)$-forms by linearity: if $\beta \in \Lambda^{p,q}(\Omega)$ is expressed in the form (1.4) of Lecture 1,

$$\bar\partial\beta = \sum_{\substack{|I|=p\\|J|=q}} \sum_{k=1}^{n} \frac{\partial \beta_{IJ}}{\partial \bar{z}_k}\, d\bar{z}_k \wedge dz^I \wedge d\bar{z}^J. \tag{3.2}$$

A straightforward computation using (3.2) shows that $\bar\partial \circ \bar\partial \equiv 0$. This leads to the *Dolbeault complex* (or $\bar\partial$-complex), for each $p \in \{0,1,\dots,n\}$,

$$\Lambda^{p,0}(\Omega) \xrightarrow{\bar\partial} \Lambda^{p,1}(\Omega) \xrightarrow{\bar\partial} \Lambda^{p,2}(\Omega) \xrightarrow{\bar\partial} \dots.$$

3.1.2. *Compatibility conditions.* The necessary conditions on $\alpha$ noted in Lecture 1 (necessary for a solution to $\bar\partial u = \alpha$ to even exist) are neatly expressed by the $\bar\partial$-complex. If $\alpha = \sum_{j=1}^n \alpha_j\, d\bar{z}_j \in \Lambda^{0,1}(\Omega)$, a straightforward computation shows

$$\bar\partial\alpha = \sum_{j<k} \left( \frac{\partial \alpha_k}{\partial \bar{z}_j} - \frac{\partial \alpha_j}{\partial \bar{z}_k} \right) d\bar{z}_j \wedge d\bar{z}_k. \qquad \textit{ExerciseI.}$$

It follows that all the equations in (1.5) of Lecture 1 are expressed by the single condition that the $(0,2)$-form $\bar\partial\alpha$ vanishes! Thus, $\bar\partial\alpha = 0$ is necessary in order to solve $\bar\partial u = \alpha$.

The same thing occurs for forms of arbitrary bi-degree. A form $\beta \in \Lambda^{p,q}(\Omega)$ satisfying $\bar\partial\beta = 0$ is called *$\bar\partial$-closed*. A necessary condition that there is a $v \in \Lambda^{p,q-1}(\Omega)$ solving $\bar\partial v = \beta$ is that $\beta$ is $\bar\partial$-closed.

3.1.3. *Volume forms.* The euclidean volume form appears in various integrals in these lectures and is denoted $dV$. In terms of the underlying real differentials,

$$dV = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Recall that $z_k = x_k + iy_k$. An elementary computation shows

$$dV = (2i)^{-n} d\bar{z}_1 \wedge dz_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_n. \qquad \textit{ExerciseI.}$$

3.2. **Subharmonic functions.** Our main focus continues to be $\Omega \subset \mathbb{C}^n$, but the complex structure plays no role in the next definition; subharmonic functions are naturally defined on domains in $\mathbb{R}^N$. For $p \in \mathbb{C}^n$ and $r \in \mathbb{R}$, let $D(p,r)$ denote the ball centered at $p$ of radius $r$. Let $bD$ denote the boundary and $\sigma(bD(p,r))$ the measure of the boundary of this ball.

**Definition 3.3.** *Let $\Omega \subset \mathbb{R}^N$. An upper semi-continuous function $g : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is* **subharmonic** *if for any $p \in \Omega$ and $r > 0$ such that $\bar{D}(p,r) \subset \Omega$, $g$ satisfies the sub-mean value property*

$$g(p) \leq \frac{1}{\sigma(bD(p,r))} \int_{bD(p,r)} f(t)\, d\sigma(t), \tag{3.4}$$

*where $d\sigma$ is induced surface measure.*

Let $SH(\Omega)$ denote the set of subharmonic functions on $\Omega$. Allowing $g$ to take the value $-\infty$ is useful so that, e.g., the function $\log|z| \in SH(\mathbb{C})$. ExerciseI.

The next result justifies the name subharmonic. Consider the following domination property on $g$:

(S) for any $p \in \Omega$ and $r > 0$ such that $\bar{D}(p,r) \subset \Omega$, if $h$ is a harmonic function on $D(p,r)$ that is continuous on $\bar{D}(p,r)$ and satisfies $h(z) \geq g(z)$ for $z \in bD(p,r)$, it also holds that $h(z) \geq g(z)$ for $z \in D(p,r)$.

**Proposition 3.5.** *An upper semi-continuous function $g : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if $g$ satisfies (S).*

3.2.1. $\Delta$ *and subharmonicity.* Verifying $g$ is subharmonic from the definition is usually not easy. A more verifiable condition holds for smooth subharmonic functions.

**Proposition 3.6.** *If $g \in C^2(\Omega)$ then $g \in SH(\Omega)$ if and only if $\Delta g(z) \geq 0$ for $z \in \Omega$*

The main part of the proof of Proposition 3.6 follows from

**Lemma 3.7.** *Suppose $g \in SH(\Omega)$ and $g \not\equiv -\infty$. Then for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$,*

$$\int_\Omega g \Delta\phi \, dV \geq 0.$$

*Proof.* To be given. ☐

*Proof of Proposition 3.6.* To be given. ☐

The differential characterization of subharmonicity shows that $SH(\Omega)$ is large. For example, if $f \in \mathcal{O}(\Omega)$ then $|f|^p \in SH(\Omega)$ for all $p > 0$ and $\log|f| \in SH(\Omega)$. ExerciseI. Also, if $h$ is harmonic on $\Omega$, then $|h|^p \in SH(\Omega)$ for all $p \geq 0$.

3.2.2. *Integrability of subharmonic functions.*

**Proposition 3.8.** *Let $\Omega \subset \mathbb{R}^N$ be a domain. If $f \in SH(\Omega)$ and $f \not\equiv -\infty$, then $f \in L^1_{loc}(\Omega)$.*

*Proof.* ExerciseII. ☐

Note that in particular Proposition 3.8 says the *polar set* of $f$, $P_f = \{x \in \Omega : f(x) = -\infty\}$ has measure zero.

3.2.3. *Bounded subharmonic functions.* There is a balance between the "lower bound of subharmonicity" of a $C^2$ function $f$ and its boundedness as a function. This plays a role later, when we discuss Hörmander and twisted $\bar\partial$ estimates in $L^2$.

**Lemma 3.9.** *The Laplacian in polar coordinates* $(r,\theta)$ *in* $\mathbb{R}^2$ *is*

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

*Proof.* ExerciseII. □

Let $D(p;a)$ denote the disc in $\mathbb{C}^1$ with center $p$ and radius $a$. Let $\mathscr{G} = \{u \in \mathrm{SH}(D) \cap C^2(D) :\ 0 \le u(z) \le 1, z \in D\}$, where $D = D(0;1)$. For $0 < \epsilon < 1$, consider the following extremal problem: how large can $K > 0$ be such that

$$\triangle u(z) \ge K \qquad \forall\, z \in D(0;\epsilon) \tag{3.10}$$

for $u \in \mathscr{G}$?

First observe it suffices to consider *radial* elements in $\mathscr{G}$.

**Lemma 3.11.** *Let* $u \in \mathrm{SH}(D) \cap C^2(D)$ *satisfy* (3.10). *There exists a radial* $v \in \mathrm{SH}(D) \cap C^2(D)$ *such that*

(i) $\|v\|_{L^\infty(D)} \le \|u\|_{L^\infty(D)}$
(ii) $\triangle v(r) \ge K$ *if* $0 \le r \le \epsilon$.

*Proof.* See McNeal-Varolin, "$L^2$ estimates for the $\bar\partial$ operator", Bull. Math Sci. 5 (2015), pages 179–249. □

Let $\mathscr{G}_{\mathrm{rad}}$ denote the radial functions in $\mathscr{G}$.

**Proposition 3.12.** *Suppose* $u \in \mathscr{G}_{rad}$, $0 < \epsilon < 1$, *and* $\triangle u(z) \ge K$ *for all* $z \in D(0;\epsilon)$. *Then*

$$K \lesssim \frac{1}{\epsilon^2}\left(\log\frac{1}{\epsilon}\right)^{-1},$$

*where the estimate* $\lesssim$ *is uniform in* $\epsilon$.

*Proof.* See McNeal-Varolin, "$L^2$ estimates for the $\bar\partial$ operator", Bull. Math Sci. 5 (2015), pages 179–249. □

*Remark* 3.13. Note that $\frac{1}{\epsilon^2}\left(\log\frac{1}{\epsilon}\right)^{-1} << \frac{1}{\epsilon^2}$ as $\epsilon \to 0$.

3.3. **Strictly plurisubharmonic functions.** The general definition of plurisubharmonicity will be given shortly. We first isolate an important subclass of these functions.

**Definition 3.14.** *If* $f \in C^2(\Omega)$, *the* **complex Hessian** *of* $f$ *is the* $n\times n$ *matrix of functions*

$$\mathcal{H}_f = \left(\frac{\partial^2 f}{\partial z_j \partial \bar z_k}\right)_{j,k=1}^n.$$

For vectors $v, w \in \mathbb{C}^n$, the complex Hessian acts as a quadratic form on the pair $(v,w)$ by the prescription

$$\mathcal{H}_f[v,\bar w] =: \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar z_k} v_j \bar w_k. \tag{3.15}$$

When the functions in $\mathcal{H}_f$ are evaluated at $z \in \Omega$, this is denoted $\mathcal{H}_f[v,\bar w](z)$.

If $f$ is *real-valued*, $\mathcal{H}_f$ is conjugate-symmetric and $\mathcal{H}_f[v,\bar v]$ is a real number for any $v \in \mathbb{C}^n$. The class of $f$ for which this number is always positive, as $z \in \Omega$ varies, is important enough to name.

**Definition 3.16.** *A $\mathbb{R}$-valued $f \in C^2(\Omega)$ is* **strictly plurisubharmonic** *at $p \in \Omega$ if*

$$\mathcal{H}_f[v, \bar{v}](p) > 0 \qquad \text{for all } v \neq 0 \in \mathbb{C}^n.$$

*If $f$ is strictly plurisubharmonic at all $p \in \Omega$, say simply that $f$ is strictly plurisubharmonic on $\Omega$.*

An easy example of a strictly plurisubharmonic function on $\mathbb{C}^n$ is $f(z) = |z|^2 = \sum_{k=1}^n z_k \bar{z}_k$. If $f$ is strictly plurisubharmonic at $p$, then $\mu = \min\{\mathcal{H}_f[v, \bar{v}](p) : |v| = 1\}$ is positive by compactness. Consequently $\mathcal{H}_f[\xi, \bar{\xi}](p) \geq \mu|\xi|^2$ for all $\xi \in \mathbb{C}^n$ by bilinearity. ExerciseI. Moreover, the continuity of $f$'s 2nd derivatives implies that for any $0 < \nu < \mu$ there is a neighborhood $U$ of $p$ such that

$$\mathcal{H}_f[\xi, \bar{\xi}](z) \geq \nu|\xi|^2 \qquad \text{for all } \xi \in \mathbb{C}^n, z \in U.$$

Thus $f$ is strictly plurisubharmonic in a neighborhood of $p$.

3.3.1. *Positive* $(1,1)$*-forms.* The linear span of the Cauchy-Riemann vector fields $\left\{\frac{\partial}{\partial z_j}\right\}$and $\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$ are denoted $T^{1,0}(\mathbb{C}^n)$ and $T^{0,1}(\mathbb{C}^n)$, respectively. A $(0,1)$-form $\alpha = \sum_{j=1}^n \alpha_j \, d\bar{z}_j$ naturally acts on $V = \sum_{j=1}^n v_j \frac{\partial}{\partial \bar{z}_j} \in T^{0,1}(\mathbb{C}^n)$ by the rule $\alpha[V] = \sum_{j=1}^n \alpha_j v_j$. Similarly for a $(1,0)$-form acting on $T^{1,0}(\mathbb{C}^n)$. The action of a $(0,1)$-form on $T^{1,0}$, resp. a $(1,0)$-form acting on $T^{0,1}$, is declared to be 0.

This duality lifts to $(p,q)$-forms acting on tensor powers of $T^{1,0}$ and $T^{0,1}$. In particular, if $\beta = \sum \beta_{jk} \, dz_j \wedge d\bar{z}_k$ is a $(1,1)$-form, set

$$\beta[V, W] = \sum_{j,k=1}^n \beta_{jk} V_j W_k \qquad \text{where } V \in T^{1,0}, W \in T^{0,1}.$$

If $V = \sum V_j \frac{\partial}{\partial z_j} \in T^{1,0}$, let $\bar{V}$ be the corresponding vector $\sum \bar{V}_j \frac{\partial}{\partial \bar{z}_j} \in T^{0,1}$. If $f \in C^2(\Omega)$, note that

$$i\partial\bar{\partial} f = i \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} \, dz_j \wedge d\bar{z}_k$$

is a *real* $(1,1)$-form since $\overline{i\partial\bar{\partial} f} = -i\bar{\partial}\partial f = i\partial\bar{\partial} f$.

**Definition 3.17** (Definition 3.16 redux)**.** *A $\mathbb{R}$-valued $f \in C^2(\Omega)$ is strictly plurisubharmonic at $p \in \Omega$ if*

$$i\partial\bar{\partial} f[V, \bar{V}](p) > 0 \qquad \text{for all } V \neq 0 \in T^{1,0}(\mathbb{C}^n).$$

3.3.2. *Useful notation for derivatives.* Some notation for derivatives and their "action" on vectors is introduced, for easy reference. If $f \in C^2(\Omega)$ and $V \in \mathbb{C}^n$ is a vector, let

$$i\partial\bar{\partial} f[V, \bar{V}] =: \sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} V_j \bar{V}_k. \tag{3.18}$$

This is consistent with the previous section if $V$ is viewed as an element of $T^{1,0}(\mathbb{C}^n)$. But it is also handy to use (3.18) when $V$ is viewed as an element of $\Lambda^{0,1}(\mathbb{C}^n)$, i.e. $V = \sum V_k \, d\bar{z}_k$. In $\mathbb{C}^n$, passage between forms and tangent vectors is done by merely "substituting" components in the two vectors space structures: since the (global) euclidean metric is used on both structures, no issues arise. There are issues on complex manifolds, with only locally defined metrics, but this is a later topic.

A notation for complex gradients is also useful: if $f \in C^1(\Omega)$ and $V \in \mathbb{C}^n$ is a vector, let

$$\partial f[V] =: \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} V_j. \tag{3.19}$$

Evaluating (3.18) and (3.19) at $z \in \Omega$ is notated $i\partial\bar{\partial} f[V,\bar{V}](z)$ and $\partial f[V](z)$.

3.4. **Plurisubharmonic functions.** A plurisubharmonic function on $\Omega \subset \mathbb{C}^n$ is defined via its restriction to complex lines. If $a, b \in \mathbb{C}^n$, the set $\ell = \{a + b\tau : \tau \in \mathbb{C}\}$ is called a complex line.

Note that not every real 2-dimensional linear subspace of $\mathbb{C}^n$ is a complex line. ExerciseI.

**Definition 3.20.** *An upper semi-continuous function $f : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$ is* **plurisubharmonic** *if for every complex line $\ell = \{a + b\tau : \tau \in \mathbb{C}\}$, the function*

$$\tau \longrightarrow f(a + b\tau)$$

*belongs to $SH(\Omega_\ell)$, where $\Omega_\ell = \{\tau \in \mathbb{C} : a + b\tau \in \Omega\}$.*

*Let $PSH(\Omega)$ denotes the set of plurisubharmonic functions on $\Omega$.*

*Remark* 3.21. It holds that $PSH(\Omega) \subset SH(\Omega)$, in general. The containment is strict as, e.g., the function $f(z_1, z_2) = |z_1|^2 - |z_2|^2$ shows. ExerciseI.

Proposition 3.6 implies a differential characterization of plurisubharmonicity.

**Proposition 3.22.** *A function $f \in C^2(\Omega) \cap PSH(\Omega)$ if and only if*

$$i\partial\bar{\partial} f[V,\bar{V}](z) \geq 0 \qquad \text{for all } V \in T^{1,0}(\mathbb{C}^n), z \in \Omega.$$

*Proof.* ExerciseII. □

A useful consequence of Proposition 3.22 is the following. Suppose $f \in C^2(\Omega)$ and $\phi \in C^2(\mathbb{R})$; let $\psi =: \phi \circ f \in C^2(\Omega)$. Then

$$i\partial\bar{\partial}\psi[V,\bar{V}](z) = \phi'(f(z)) \cdot i\partial\bar{\partial} f[V,\bar{V}](z) + \phi''(f(z)) \cdot |\partial f[V](z)|^2. \tag{3.23}$$

ExerciseI. This is easy, but good practice with the notation. It follows from (3.23) that if $f \in C^2(\Omega) \cap PSH(\Omega)$ and $\phi \in C^2(\mathbb{R})$ is a convex, increasing function, then $\phi \circ f \in PSH(\Omega)$.

3.4.1. *Basic approximation of $PSH(\Omega)$.* Approximation of a general plurisubharmonic function by more regular plurisubharmonic functions has several variants and many applications. The next result is one of the most basic. It is used to extend the scope of weighted $L^2$ estimates for $\bar{\partial}$ in Lecture 7. Demailly's lemma is another such approximation result; see Demailly, "Regularization of closed positive currents and intersection theory", J. Alg. Geom. (1992), pages 361–409.

**Proposition 3.24.** *Let $\Omega \subset \mathbb{C}^n$ and $\Omega_j = \left\{z \in \Omega : dist(z, b\Omega) > \frac{1}{j} \text{ and } |z| < j\right\}$ for $j \in \mathbb{Z}^+$.*

*If $f \in PSH(\Omega)$, there exists a family of functions $f_j$ such that*

(i) $f_j \in C^\infty(\Omega)$,
(ii) $f_j(z) \geq f_{j+1}(z)$,
(iii) *$f_j$ is strictly plurisubharmonic on $\Omega_j$, and*
(iv) $\lim_{j\to\infty} f_j(z) = f(z)$ *for each $z \in \Omega$.*

*Proof.* Let $\phi \in C_0^\infty(B(0,1))$ be chosen with the properties

(i) $\phi \geq 0$,
(ii) $\phi$ is *radial*, i.e., $\phi(a) = \phi(b)$ when $|a| = |b|$, and
(iii) $\int \phi \, dV = 1$.

Define

$$v_j(z) = \int_{\Omega_j} f(\zeta)\, \phi\left(j(z-\zeta)\right) j^{2n}\, dV(\zeta).$$

Since Proposition 3.8 implies $f \in L^1(\Omega_j)$, $v_j$ is well-defined; differentiation under the integral shows $v_j \in C^\infty$. For $z \in D_j$, the obvious linear change of variables gives

$$v_j(z) = \int_{|\zeta|<1} f\left(z - \frac{\zeta}{j}\right) \phi(\zeta)\, dV(\zeta). \tag{3.25}$$

We now claim that

(a) $v_j \in PSH(\Omega_j)$,
(b) $v_j(z) \geq v_{j+1}(z)$, and
(c) $\lim_{j\to\infty} v_j(z) = f(z)$ for each $z \in \Omega$.

This suffices since, defining $f_j(z) = v_j(z) + \frac{1}{j}\,|z|^2$, shows $f_j$ satisfies (a)-(c) if $v_j$ does and

$$\mathcal{H}f_j[\xi,\xi] = \mathcal{H}v_j[\xi,\xi] + \frac{1}{j}\,|\xi|^2 \geq \frac{1}{j}|\xi|^2.$$

Property (a) follows from the corresponding property for $f$: for small $r > 0$

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} v_j\left(a + re^{i\theta}w\right)\, d\theta &= \int_{|\zeta|<1}\left[\frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}w - \frac{\zeta}{j}\right)\, d\theta\right]\phi(\zeta)\, dV(\zeta)\\
&\geq \int_{|\zeta|<1} f\left(a - \frac{\zeta}{j}\right)\phi(\zeta)\, dV(\zeta)\\
&= v_j(a).
\end{aligned}$$

Properties (b) and (c) follow in a very similar fashion. See Range, "Holomorphic functions and integral representations in Several Complex Variables", Theorem 4.12, for details. □

## References