MSRI2018, LECTURE 5

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READ WITH CAUTION, TYPOS ABOUND.

5. Pseudoconvexity

Domains with regular (smooth) boundary are studied first. This has a great computational advantage, as well as a conceptional one. Results on domains with less boundary regularity can often be deduced from their smooth approximants.

5.1. Defining functions.

Let $\mathbb{R}^N$ denote N-dimensional real euclidean space and $(a_1, \dots, a_N)$ denote the standard coordinates on $\mathbb{R}^N$. As with subharmonic functions, several notions are most naturally stated in $\mathbb{R}^N$.

Definition 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain. A real-valued function $r \in C^1\left(\overline{\Omega}\right)$ is a* **defining function for** Ω *if*

$$\Omega = \left\{a \in \mathbb{R}^N : r(a) < 0\right\} \quad \textit{and} \quad dr \neq 0 \textit{ on } \left\{a \in \mathbb{R}^N : r(a) = 0\right\} =: b\Omega.$$

Higher levels of regularity for $b\Omega$ are prescribed by requiring r to belong to smoother functions spaces, e.g. $C^k\left(\overline{\Omega}\right)$, $C^\infty\left(\overline{\Omega}\right)$, $C^\omega\left(\overline{\Omega}\right)$. When r can be chosen in $C^\infty\left(\overline{\Omega}\right)$, $b\Omega$ is called *smooth* and Ω is called a *smoothly bounded* domain.

Defining functions are **not** unique. However if r is a fixed defining function and ρ is any other defining function for Ω, there exists $h \in C^1\left(\overline{\Omega}\right)$ with $h > 0$ in a neighborhood of $b\Omega$ such that $\rho = h \cdot r$. ExerciseII. When $b\Omega$ has higher than C^1 regularity, h is in the corresponding function space.

Another common definition for Ω to have C^k boundary is to require that $b\Omega$ be a C^k manifold. If Ω has C^k boundary in this sense, it is not difficult to construct a C^k defining function – in fact, infinitely many – in the sense of Definition 5.1.

One useful choice of defining function involves the euclidean structure of $\mathbb{R}^N$. If $F \subset \mathbb{R}^N$ is closed, let $d_F(a) = \inf\{|a - f| : f \in F\}$. The *signed distance function to* $b\Omega$ is defined

$$\delta_{b\Omega}(a) = \begin{cases} -d_{b\Omega}(a), & a \in \Omega \\ \ \ d_{b\Omega}(a), & a \in \Omega^c \end{cases}$$

If $b\Omega$ is a C^k manifold, $\delta_{b\Omega}$ is a C^k defining function. ExerciseIII.

5.2. Tangent spaces to $b\Omega$.

Definition 5.2. *Let $\Omega \subset \mathbb{R}^N$ be a domain with C^1 boundary, $p \in b\Omega$, and r a defining function for Ω.*

The (real) **tangent space to $b\Omega$ at p is**

$$T_p(b\Omega) = \left\{ u = (u_1, \dots u_N) \in \mathbb{R}^N : \sum_{j=1}^N \frac{\partial r}{\partial a_j}(p) u_j = 0 \right\}.$$

Note that $T_p(b\Omega)$ is a vector space, has dimension $N-1$, and does not depend on the choice of defining function r. ExerciseI.

Recall the standard coordinates on $\mathbb{C}^n$ are being denoted $z_j = x_j + iy_j$, $j = 1, \dots n$. If Ω is a domain in $\mathbb{C}^n$ (in particular $N = 2n$ above), writing Definition 5.2 in terms of z_j derivatives is both useful and enlightening. First, if the underlying structure on $\mathbb{R}^{2n}$ is given with the order $(x_1, y_1, \dots, x_n, y_n)$, then a vector $(u_1, v_1, \dots, u_n, v_n) \in \mathbb{R}^{2n}$ belongs to $T_p(b\Omega)$ if

$$\sum_{j=1}^{n} \frac{\partial r}{\partial x_j}(p)u_j + \sum_{j=1}^{n} \frac{\partial r}{\partial y_j}(p)v_j = 0. \tag{5.3}$$

Let $w_j = u_j + iv_j$ to express vectors using the structure of $\mathbb{C}^n$. It follows that

$$(5.3) \Leftrightarrow \mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)w_j\right) = 0 \qquad \mathit{ExerciseI}.$$

Continue letting $\Omega \subset \mathbb{C}^n$ and use complex notation. Note that $w = (w_1, \dots, w_n) \in T_p(b\Omega)$ does **not** necessarily force $iw \in T_p(b\Omega)$, i.e., $T_p(b\Omega)$ is not a *complex* vector space. A distinguished subspace of $T_p(b\Omega)$ plays a large role in the sequel.

Definition 5.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^1 boundary, $p \in b\Omega$, and r a defining function for Ω. The* **complex tangent space to $b\Omega$ at p is**

$$T_p^{\mathbb{C}}(b\Omega) = \left\{ w \in \mathbb{C}^n : \sum_{j=1}^{N} \frac{\partial r}{\partial z_j}(p)w_j = 0 \right\}.$$

Elementary arguments show $T_p^{\mathbb{C}}(b\Omega) \subset T_p(b\Omega)$ is the maximal complex subspace and $\dim_{\mathbb{R}} T_p^{\mathbb{C}}(b\Omega) = 2n-2$. ExerciseI.

The single "direction" (from the vector space point of view) $X \in T_p(b\Omega) \setminus T_p^{\mathbb{C}}(b\Omega)$ is often called the *bad tangential direction* to $b\Omega$ at p. The definite article is a little mis-leading: a generic tangential vector to $b\Omega$ has a non-zero component in the bad direction.

5.3. Convex domains.

Definition 5.5. *Let $U \subset \mathbb{R}^N$ be open and $f \in C^2(U)$. The* **Hessian of f** *at $x \in U$ is the $N \times N$ matrix of functions*

$$H_f = \left(\frac{\partial^2 f}{\partial a_j \partial a_k}\right)_{j,k=1}^{N}.$$

For vectors $v, w \in \mathbb{R}^N$, the Hessian acts on the pair (v, w) by the prescription

$$H_f[v, w] =: \sum_{j,k=1}^{N} \frac{\partial^2 f}{\partial a_j \partial a_k} v_j w_k. \tag{5.6}$$

Evaluation at $x \in U$ is denoted $H_f[v, w](x)$.

A *function* $f \in C^2(U)$ is convex near $x \in U$ – in the usual sense that $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for all a, b close to x and for all $\lambda \in [0, 1]$ – if and only if $H_f[w, w](y) \geq 0$ for all y close to x and for all $v \in \mathbb{R}^N \setminus \{0\}$. A similar condition turns out to characterize convex *sets* with sufficiently smooth boundaries in $\mathbb{R}^N$:

Definition 5.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain with C^2 boundary and r be a defining function for Ω. Say that Ω is* **convex** *at $p \in b\Omega$ if*

$$H_r[v, v](p) \geq 0 \qquad \text{for all } v \in T_p(b\Omega). \tag{5.8}$$

Say that Ω is **strongly convex** *at p if inequality (5.8) is strict for $v \neq 0$.*

The usual definition for $\Omega \subset \mathbb{R}^N$ to be convex is geometric: if $x, y \in \Omega$, then $\lambda x + (1-\lambda)y \in \Omega$ for all $\lambda \in [0,1]$. It is not obvious Definition 5.7 coincides with this definition, but this indeed holds. ExerciseII.

Inequality (5.8) does not depend on the choice of defining function. ExerciseI. Also note that semi-definiteness in (5.8) is only being prescribed in directions $v \in T_p(b\Omega)$. The function $r(x,y) = y - y^2 + x^2$ near $(0,0) \in \mathbb{R}^2$ locally defines a convex domain, but shows the Hessian of a general defining function may be negative in non-tangential directions. It is however possible to *choose* a defining function for any convex domain which is fully convex, see [**?**].

5.4. **Pseudoconvexity.** Now let $\Omega \subset \mathbb{C}^n$ with C^2 boundary. For complex analysis, the principal convexity notion on Ω has a parallel formulation to Definition 5.7.

Definition 5.9. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^2 boundary and r be a defining function for Ω. Say that Ω is* **Levi pseudoconvex** *at $p \in b\Omega$ if*

$$\mathcal{H}_r[\xi, \bar{\xi}](p) \geq 0 \qquad \text{for all } \xi \in T_p^{\mathbb{C}}(b\Omega). \tag{5.10}$$

Say that Ω is **strongly pseudoconvex** *at p if inequality (5.10) is strict for $\xi \neq 0$.*

Re-expressing Levi pseudodconvexity using notation (3.18) of Lecture 3, Ω satisfies Definition 5.9 if

$$i\partial\bar{\partial} r[\xi, \bar{\xi}](p) \geq 0 \qquad \text{for all } \xi \in T_p^{\mathbb{C}}(b\Omega) \text{ and for all } p \in b\Omega, \tag{5.11}$$

for any defining function r for Ω. Condition (5.11) is independent of r ExerciseI.

Some basic facts about pseudoconvexity are given in the next three Propositions.

Proposition 5.12. *Any smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$ is Levi pseudoconvex.*

Proof. ExerciseII (Sketch) Use Taylor's theorem. First show that

$$\frac{1}{2} H_r[v,v] = \mathcal{H}_r[\xi,\xi] + \text{Re } \mathcal{Q}_r[\xi,\xi]$$

where $\xi_j = v_{2j-1} + iv_{2j}$ and

$$\mathcal{Q}_r[\xi,\xi] = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k} \xi_j \xi_j.$$

Apply this to ξ and $i\xi$, if $\xi \in T_p^{\mathbb{C}}(b\Omega)$. □

Converse of this proposition is not true as, e.g., any smoothly bounded non-convex $\Omega \subset \mathbb{C}$ shows.

Proposition 5.13. *Suppose $\Omega \subset \mathbb{C}^n$ with C^2 boundary, $p \in b\Omega$, U is a neighborhood of p, and $w = F(z)$ is a biholomorphic map. Let $\tilde{\Omega} = F(U \cap \Omega)$.*

Then

$$\tilde{\Omega} \textit{ is Levi pseudoconvex at } q = F(p) \Leftrightarrow \Omega \textit{ is Levi pseudoconvex at } p.$$

Proof. (Sketch) Check $\rho =: r \circ F^{-1}$ is a C^2 defining function for $\tilde{\Omega}$ near q. Show

$$\partial r[\xi](p) = \partial\rho\left[F'(p)\xi\right](q).$$

Then show $i\partial\bar{\partial} r[\xi, \bar{\xi}](p) = i\partial\bar{\partial}\rho\left[F'(p)\xi, \overline{F'(p)\xi}\right](q)$. □

Proposition 5.14. *Suppose $\Omega \subset \mathbb{C}^n$ with C^2 boundary and is Levi pseudoconvex near $p \in b\Omega$.*

Then there exists a defining function ρ on a neighborhood U of p such that for all $q \in U$

$$i\partial\bar{\partial}\rho[\xi, \bar{\xi}](q) \geq 0 \qquad \textit{for all } \partial\rho[\xi](q) = 0.$$

The level set $b\Omega^q =: \{z \in U : r(z) = r(q)$ is a hypersurface approximating $b\Omega$ near p. The condition $\partial\rho[\xi](q) = 0$ says that $\xi \in T_q^{\mathbb{C}}(b\Omega^q)$; thus Proposition 5.14 says these level sets locally bound pseudoconvex domains.

Proposition 5.14 only holds *locally.* The Diederich-Fornæss worm is a counterexample to the global statement (from the outside).

5.4.1. *Strongly pseudoconvex domains.* Strong pseudoconvexity is much more malleable than pseudoconvexity. The next three results are fundamental.

Proposition 5.15. *Let $\Omega \subset \mathbb{C}^n$ with C^2 boundary be strongly pseudoconvex near $p \in b\Omega$.*

Then there exists a neighborhood U of p, a defining function ρ for $\Omega \cap U$, and a constant $c > 0$ such that

$$i\partial\bar{\partial}\rho[\xi, \bar{\xi}](q) \geq c|\xi|^2 \qquad \textit{for all } \xi \in \mathbb{C}^n.$$

The gain in Proposition 5.15 is that the form $i\partial\bar{\partial}\rho$ is positive definite in *all* directions ξ and for all $q \in U$, not only for $\xi \in T_p^{\mathbb{C}}(b\Omega)$ and $p \in b\Omega$.

Remark 5.16. If every $p \in b\Omega$ is strongly pseudoconvex, the local defining functions in 5.15 can be patched together: there is a neighborhood $\mathscr{C}$ of $b\Omega$ and a defining function ρ on $\Omega \cap \mathscr{C}$ such that $i\partial\bar{\partial}\rho[\xi, \bar{\xi}](z) \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^n$ and $z \in \mathscr{C}$. ExerciseI.

Proof. To be given. ☐

Proposition 5.17. *Let $\Omega \subset \mathbb{C}^n$ with C^2 boundary be strongly pseudoconvex near $p \in b\Omega$.*

Then there exists holomorphic coordinates $w = (w_1, \dots, w_n)$ in a neighborhood U of p such that $\Omega \cap U$ is convex in the w coordinates.

Proof. To be given. ☐

Proposition 5.17 is local. In general, a strongly pseudoconvex domain is **not** biholomorphically equivalent to a convex domain. Give example of torus??

The last basic result on strongly pseudoconvex domains involves the notion of an exhaustion function. If $D \subset \mathbb{C}^n$ is an open set (no boundary regularity needed), a function $g : D \to \mathbb{R}$ is an **exhaustion function** if for every $\mu \in \mathbb{R}$, the set $D_\mu = \{z \in D : g(z) < \mu\}$ is relatively compact in D. An exhaustion function necessarily satisfies $g(z) \to \infty$ as $z \to bD$, and this condition is also sufficient if D is bounded.

Proposition 5.18. *If $\Omega \subset\subset \mathbb{C}^n$ is Levi strongly pseudoconvex, then Ω admits a* **strongly plurisubharmonic** *exhaustion function.*

Proof. (Sketch) From Remark 5.16, choose a neighborhood $\mathscr{C}$ of $b\Omega$ and a strictly plurisubharmonic r on $\Omega \cap \mathscr{C}$ such that $\Omega \cap \mathscr{C} = \{z \in \mathscr{C} : r(z) < 0\}$ and $dr(z) \neq 0$ for $z \in \mathscr{C}$.

WLOG, we can assume $r \in C^2\left(\overline{\Omega}\right)$ (why?). Subtracting a large positive constant from r, can also assume $r < 0$ on all of Ω – note use of $\Omega \subset\subset \mathbb{C}^n$.

Direct computation shows $\phi =: -\log(-r)$ is strictly plusrisubharmonic of $\Omega \cap \mathscr{C}$ ExerciseI. Let

$$\mu =: \inf\left\{\mathcal{H}_\phi[\xi, \bar{\xi}](z) : z \in \Omega \setminus \mathscr{C} \text{ and } |\xi| = 1\right\}.$$

Then $\mu > -\infty$. The function $\psi =: \phi + (\mu + 1)|z|^2$ works. ☐

Proposition 5.18 implies that strongly pseudoconvex domains Ω can be approximated from the inside and the outside by strongly pseudoconvex domains. This is a fundamental geometric fact about strongly pseudoconvex domains which does *not* hold on more general pseudoconvex domains. To record the state of affairs, first

Definition 5.19. *A collection* $\{U_\alpha\} \subset \mathbb{C}^n$ *of open sets is a* **neighborhood basis** *of some set* $D \subset \mathbb{C}^n$ *if* $\cap_\alpha U_\alpha = \overline{D}$.

If $\Omega \subset \mathbb{C}^n$ *is a domain, the* **Nebenhülle of** Ω *is defined*

$$\mathscr{N}(\Omega) = \text{ interior } \left\{\cap U_\alpha : U_\alpha \text{ is psesudoconvex and } \overline{\Omega} \subset U_\alpha\right\}.$$

If $\mathscr{N}(\Omega) \setminus \Omega$ *has interior points, then* Ω *is said to have nontrivial Nebenhülle.*

Sard's theorem and Proposition 5.18 imply

Proposition 5.20. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded, strongly pseudoconvex domain. There exist montone sequences of strongly pseudoconvex domains* $\{I_j\} \subset \Omega$ *and* $\{O_j\} \supset \overline{\Omega}$,

$$I_j \subset I_{j+1} \quad \text{and} \quad O_j \supset O_{j+1} \text{ for all } j,$$

such that

- (i) $\Omega = \cup I_j$ *and*
- (ii) O_j *is a neighborhood basis for* Ω.

In particular, $\mathscr{N}(\Omega) = \emptyset$.

Proof. To be discussed. ☐

5.4.2. *Weakly pseudoconvex domains.* Consider the propositions in the previous section on weakly pseudoconvex domains (still with smooth boundary). The first two observations are negative results.

Proposition 5.21 (Non-Proposition 5.15)**.** *There exist smoothly bounded pseudoconvex* $\Omega \subset \mathbb{C}^n$, $p \in b\Omega$, *and* U *a neighborhood of* p, *for which* **every** *local defining function* r *for* $U \cap \Omega$ *has the property that* $i\partial\bar{\partial}r[\nu,\nu](q) < 0$ *if* $\nu \notin T_q^{\mathbb{C}}(b\Omega)$ *and* $q \in U$.

Give reference to Fornaess example.

Proposition 5.22 (Non-Proposition 5.17)**.** *There exist smoothly bounded pseudoconvex* $\Omega \subset \mathbb{C}^n$ *and* $p \in b\Omega$ *such that for* **no** *neighborhood* U *does there exist a holomorphic coordinate system on* U *in which* $\Omega \cap U$ *is convex.*

Originally, Proposition 5.22 was discovered by Kohn-Nirenberg in the early 1970s. Simplifications are due to Fornæss. The Kohn-Nirenberg proof is discussed in Problem sets. For so-called "type 4" boundary points, issues remain unresolved.

A positive result, however, holds relative to 5.18:

Proposition 5.23. *If* $\Omega \subset\subset \mathbb{C}^n$ *is a Levi pseudoconvex domain with* C^3 *boundary, then* Ω *admits a strongly plurisubharmonic exhaustion function.*

Proof. To be given. ☐

This result yields half of Proposition 5.20.

Proposition 5.24. *If* $\Omega \subset\subset \mathbb{C}^n$ *is a Levi pseudoconvex domain with* C^3 *boundary, then there exists an increasing sequence of strongly pseudoconvex domains* $\{\Omega_j\}$, $\Omega_j \subset \Omega$, *such that* $\Omega = \cup_{j=1}^\infty \Omega_j$.

Both Propositions 5.23 and 5.24 hold if $b\Omega$ is only C^2, but the proofs are more difficult.

However the other half of Proposition 5.20 – approximating Ω by pseudoconvex domains from the outside – does not hold on a general weakly pseudoconvex domain Ω. This was originally discovered by Diederich-Fornæss. Their domains W are known as "worm domains" and will be discussed in Problem sets, as well as the non-smoothly bounded Hartogs triangles which underlie the Diederich-Fornaess construction.