

MSRI2018, LECTURE 12

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1. THE CARTAN-THULLEN THEORY

The Cartan-Thullen theory deals with characterizing domains of holomorphy. It shows that every weak domain of holomorphy is a domain of holomorphy. **In this lecture, we will finish the proof of the Levi problem for smoothly bounded domains.**

1.1. Introduction. Let us recall some definitions:

Definition 1.1. A holomorphic function f on a domain $\Omega \in \mathbb{C}^n$ is said to be **completely singular** at $p \in b\Omega$ if for every connected neighborhood U of p there is no holomorphic function $h \in \mathcal{O}(U)$ which agrees with f on some connected component of $U \cap \Omega$, having p on the boundary.

Definition 1.2. A domain $\Omega \subset \mathbb{C}^n$ is said to be a **weak domain of holomorphy** if for every $p \in b\Omega$ there is a $f_p \in \mathcal{O}(\Omega)$ which is completely singular at p . If there is a holomorphic function $f \in \mathcal{O}(\Omega)$ which is completely singular at every boundary point $p \in b\Omega$, then we say that Ω is a **domain of holomorphy**.

1.2. Holomorphic Convexity. One important tool that will help us achieve our goal is holomorphic convexity.

Definition 1.3. A domain $\Omega \subset \mathbb{C}^n$ is called **holomorphically convex** if for every compact subset $K \subset \Omega$, the holomorphically convex hull of K ,

$$\widehat{K}_{\mathcal{O}(\Omega)} = \{z \in \Omega : |f(z)| \leq |f|_K \text{ for all } f \in \mathcal{O}(\Omega)\}$$

is relatively compact in Ω . We will denote this hull by \widehat{K} when the domain is understood.

Let us look at some examples to familiarize ourselves with this concept:

- (1) Let $\Omega = \mathbb{C}$ and $K = \{z \in \mathbb{C} : |z| = 1\}$. Then $\widehat{K}_{\mathcal{O}(\mathbb{C})} = \{z \in \mathbb{C} : |z| \leq 1\}$.
- (2) Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 1 < |z_1|^2 + |z_2|^2 < 4\}$ and consider the compact subset $K = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 2\} \subset \Omega$. Then

$$\widehat{K}_{\mathcal{O}(\Omega)} = \{(z_1, z_2) \in \mathbb{C}^2 : 1 < |z_1|^2 + |z_2|^2 \leq 2\}$$

which is a closed subset of Ω but fails to be relatively compact in Ω . Hence Ω is not holomorphically convex.

Exercise I: Check that $\widehat{\widehat{K}} = \widehat{K}$.

Exercise I: If $K \subset \Omega$ is compact, then \widehat{K} is closed in Ω .

Lemma 1.4. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $K \subset \Omega$ be compact. Then given $M, \epsilon > 0$, and $p \in \Omega \setminus \widehat{K}$, there is $f \in \mathcal{O}(\Omega)$ with $|f|_K < \epsilon$ and $|f(p)| > M$.

Proof. Since $p \notin \widehat{K}$, there is $h \in \mathcal{O}(\Omega)$ with $|h|_K < |h(p)|$. After multiplying by suitable constant, we can assume that $|h|_K < 1 < |h(p)|$; now take $f = h^l$ with sufficiently large l . \square

Lemma 1.5. *Let $\Omega \subset \mathbb{C}^n$ be holomorphically convex. Then there is a sequence of compact subsets $\{K_j\}$ of Ω satisfying the following:*

- (a) $K_j \subset\subset \text{int } K_{j+1}$ for each j and $\bigcup_{j=1}^{\infty} K_j = \Omega$,
 (b) each K_j is $\mathcal{O}(\Omega)$ -convex.

Remark 1.6. A collection of compact sets satisfying (a) in the above lemma is called a **normal exhaustion** of Ω .

Proof. It is easy to construct a normal exhaustion $\{Q_j\}$ of Ω **Exercise I**. Since Ω is holomorphically convex, \widehat{Q}_j is compact for all j . Set $K_1 = \widehat{Q}_1$. Then K_1 is compact and $\widehat{K}_1 = K_1$. Suppose K_1, \dots, K_m with the desired properties have already been constructed. Choose $j_m > m$ such that $K_m \subset \text{int } K_{j_m}$, and set $K_{m+1} = \widehat{Q}_{j_m}$. Then the sequence $\{K_m\}_{m=1}^{\infty}$ satisfies (a) and (b). \square

1.3. Pseudoconvex domains are holomorphically convex. Here we will use the tools which are used in Cartan-Thullen theory to show that weak domains of holomorphy are holomorphically convex. In fact, the converse is also true, i.e., holomorphically convex domains are weak domain of holomorphy.

Lemma 1.7. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then for every $p \in b\Omega$, there is $f_p \in \mathcal{O}(\Omega)$ such that $\lim_{\substack{z \in \Omega \\ z \rightarrow p}} |f_p(z)| = \infty$.*

Proof. This follows from Theorem (1.5) **check the number for Solution of Levi problem** in Lecture 10, the solution of Levi problem. \square

Theorem 1.8. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then Ω is holomorphically convex.*

Proof. Refer to Theorem 3.14 of Range's book [1], page 73. \square

1.4. Solution to Levi problem for smoothly bounded domains. In Lecture 10, we already proved that a smoothly bounded pseudoconvex domain is a weak domain of holomorphy. Now we will show that it is in fact a domain of holomorphy. For that, it suffices to show that every holomorphically convex domain in \mathbb{C}^n is a domain of holomorphy.

Lemma 1.9. *Let $\{K_j\}$ be a normal exhaustion of the domain Ω . Then there are a subsequence $\{j_m\}$ of \mathbb{N} and a sequence $\{p_m\}$ of points in Ω such that*

$$p_m \in K_{j_{m+1}} \setminus K_{j_m} \quad \text{for } j = 1, 2, \dots, \quad (1.10)$$

and

$$\begin{aligned} &\text{for every } p \in b\Omega \text{ and every connected neighborhood } U \\ &\text{of } p, \text{ each component } D \text{ of } U \cap \Omega \text{ contains infinitely} \\ &\text{many points from } \{p_m\}. \end{aligned} \quad (1.11)$$

Proof. Let $\{a_j : j = 1, 2, \dots\}$ be an enumeration of the points in Ω with rational coordinates. Let $r_j = \text{dist}(a_j, b\Omega) < \infty$; then $B_j = B(a_j, r_j)$ is contained in Ω . Let $\{Q_m : m = 1, 2, \dots\}$ be a sequence of such balls B_j which contains each B_j infinitely many times; for example, we may choose the sequence $B_1, B_1, B_2, B_1, B_2, B_3, B_1, \dots$. Now take $K_{j_1} = K_1$. Proceeding inductively, assume that $l > 1$ and p_1, \dots, p_{l-1} and K_{j_1}, \dots, K_{j_l} have been found so that (1.10) holds for $m = 1, \dots, l-1$; since Q_1 is not contained in any compact subset of Ω , we may choose $p_l \in Q_l \setminus K_{j_l}$ and then j_{l+1} so that $p_l \in K_{j_{l+1}}$. Then (1.10) holds for all $m = 1, 2, \dots$. We now verify that $\{p_m\}$ satisfies (1.11). Since D is a component of the open set $U \cap \Omega$, D is open (in \mathbb{C}^n) and closed in $U \cap \Omega$. Since U is connected and $D \neq U$, D

cannot be closed in U . Hence there is a point $q \in (bD \cap U) \setminus D$. Since $D \subset \Omega$ and D is closed in $U \cap \Omega$, we must have $q \in bD \cap U \cap b\Omega$. Thus there is $a_k \in D$ with rational coordinates sufficiently close to q , so that $B_k \in D$. Since B_k occur infinitely many time in the sequence $\{Q_m\}$, and $p_m \in Q_m$ for $m = 1, 2, \dots$, B_k contains infinitely many points of $\{p_m\}$. \square

Theorem 1.12. *A holomorphically convex domain Ω in \mathbb{C}^n is a domain of holomorphy.*

Proof. Using Lemma (1.5), we get a normal exhaustion of Ω by $\mathcal{O}(\Omega)$ -convex sets, say $\{K_j\}$. We apply Lemma (1.9) to this exhaustion to obtain a sequence $\{p_m\}$ satisfying (1.10) and (1.11). We now construct $f \in \mathcal{O}(\Omega)$ which is completely singular at every point in $b\Omega$. Passing through a subsequence, we can assume that for each m , $p_m \in K_{m+1} \setminus K_m$, we apply Lemma (1.4) to obtain functions $f_j \in \mathcal{O}(\Omega)$ satisfying

$$|f_j|_{K_j} < 2^{-j}, \quad j = 1, 2, \dots \quad (1.13)$$

and

$$|f_m(p_m)| > m + 1 + \sum_{j=1}^{m-1} |f_j(p_m)|, \quad m = 2, 3, \dots \quad (1.14)$$

From (1.13), $f = \sum f_j$ converges compactly in Ω , hence $f \in \mathcal{O}(\Omega)$, and (1.14) implies

$$|f(p_m)| \geq |f_m(p_m)| - \sum_{j \neq m} |f_j(p_m)| > m + 1 - \sum_{j > m} |f_j(p_m)| \quad \text{for } m \geq 2.$$

It then follows from (1.13) that $\sum_{j > m} |f_j(p_m)| < \sum 2^{-j} \leq 1$, and hence $|f(p_m)| > m$. This means $\lim_{m \rightarrow \infty} |f(p_m)| = \infty$.

We will show that f is completely singular at every $p \in b\Omega$. If U is a component of $D \cap \Omega$, where U is a connected neighborhood of p , suppose there is $h \in \mathcal{O}(U)$ with $f|_D = h|_D$. After replacing U by $U'(p) \subset\subset U$ and D by a component D' of $U' \cap \Omega$ which meets D , we may assume that $|h|_{D'} \leq |h|_{U'} < \infty$. Hence f would have to be bounded on D' , and this contradicts (1.11) and $\lim_{m \rightarrow \infty} |f(p_m)| = \infty$. \square

REFERENCES

- [1] Range, R. Michael. **Holomorphic Functions and Integral Representation in Several Complex Variables**. Springer-Verlag.