MSRI 2018: THE $\overline{\partial}$-PROBLEM IN THE TWENTY-FIRST CENTURY

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Homework-6

Definition: The Bergman projection $\boldsymbol{B}_\Omega = \boldsymbol{B}$ is the Hilbert space orthogonal projection from $L^2(\Omega) \to A^2(\Omega)$. It is achieved by integrating against the Bergman kernel, i.e., for $f \in L^2(\Omega)$,

$$\boldsymbol{B}(f)(z) = \int_\Omega B(z,w) f(w)\, dV(w).$$

(1) *Biholomorphic Transformation Law:* This exercise shows that a biholomorphism $F : \Omega \to \widetilde{\Omega}$ induces an isomorphism of the corresponding Bergman spaces, and allows us to deduce a transformation law for the Bergman kernel.

Recall $J_{\mathbb{C}}F$ denotes the $n \times n$ matrix whose entries are the holomorphic partial derivatives of the components of F. The determinant $\det J_{\mathbb{C}}F$ is called the *complex Jacobian*. Also, $J_{\mathbb{R}}F$ denotes the real $2n \times 2n$ matrix of partial derivatives which arises by thinking of F as a map from $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. The determinant $\det J_{\mathbb{R}}F$ is called the *real Jacobian*.

(a) Prove that $\det J_{\mathbb{R}}F = |\det J_{\mathbb{C}}F|^2$.

(b) Let $F : \Omega \to \widetilde{\Omega}$ be a biholomorphism and $\{\widetilde{\phi}_\alpha\}_{\alpha \in \mathcal{A}}$ be an orthonormal basis for $A^2(\widetilde{\Omega})$. Prove F induces an orthonormal basis $\{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ for $A^2(\Omega)$, where

$$\phi_\alpha := \det J_{\mathbb{C}}F \cdot \widetilde{\phi}_\alpha \circ F. \tag{6.1}$$

(c) Prove that

$$B_\Omega(z,w) = \det J_{\mathbb{C}}F(z) \cdot B_{\widetilde{\Omega}}(F(z), F(w)) \cdot \overline{\det J_{\mathbb{C}}F(w)}. \tag{6.2}$$

(2) *Ramadanov's theorem:* Let $\{\Omega_j\}_{j \in \mathbb{Z}^+}$ be a set of domains which monotonically exhaust a bounded domain $\Omega \subseteq \mathbb{C}^n$ from inside, and let B_j and B_Ω denote the Bergman kernels of Ω_j and Ω, respectively. Show that $B_j(z,w) \to B_\Omega(z,w)$ uniformly on compact subsets of $\Omega \times \Omega$.

(3) *Explicit Bergman computations:* Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$.

(a) Find an orthonormal basis for $A^2(\mathbb{D})$.

(b) Show that the Bergman the Bergman kernel on $\mathbb{D}$ is given by

$$B(z,w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

(c) Let f be an *antiholomorphic* $L^2(\mathbb{D})$ function, i.e., $\overline{f} \in A^2(\mathbb{D})$. Show that its Bergman projection $\boldsymbol{B}(f) = f(0)$.

(d) Show that the Bergman Kernel of the unit ball in $\mathbb{C}^n$ is given by

$$B(z,w) = \frac{n!}{\pi^n (1 - z\bar{w})^{n+1}}.$$

(4) *Diagonal boundary behavior:* Let $\delta_\Omega(z) = \delta(z)$ denote the distance to the boundary function of the domain

(a) Let $\Omega \subset \mathbb{C}^n$ be a *smoothly bounded* domain. Prove that $B_\Omega(z,z) \lesssim \delta(z)^{-n-1}$

(b) Let $\Omega \subset \mathbb{C}^n$ be *any* domain, with no assumption on boundary regularity. Prove that $B_\Omega(z,z) \lesssim \delta(z)^{-2n}$.

(c) Give an example of a domain $\Omega \subset \mathbb{C}^n$ with a *non-smooth* boundary point p, and a path of points $z \to p$ such that $B_\Omega(z,z) \approx \delta(z)^{-2n}$ for all points along this path.

(5) L^p *mapping range:* By virtue of the fact that the $\boldsymbol{B}$ is *defined* as an orthogonal projection on a Hilbert space, it is an L^2 bounded operator. It is a very important question to understand how $\boldsymbol{B}$ acts on other L^p spaces. In the case of the unit disc $\mathbb{D} \subset \mathbb{C}$, we will show that $\boldsymbol{B}$ is a bounded operator from $L^p(\mathbb{D}) \to A^p(\mathbb{D})$ for all $1 < p < \infty$.

(a) *Young's Test:* This is a standard tool to show the L^p boundedness of an integral operator.
Let $\Omega \subseteq \mathbb{C}^n$ be a domain, K be an a.e. positive, measurable function on $\Omega \times \Omega$, and $\mathcal{K}$ be the integral operator with kernel K. Suppose that there exists a fixed constant C, such that

$$\int_\Omega K(z,w)\, dV(w) \leq C, \qquad \forall z \in \Omega$$
$$\int_\Omega K(z,w)\, dV(z) \leq C, \qquad \forall w \in \Omega.$$

Then $\mathcal{K}$ is a bounded operator on $L^p(\Omega)$, for all $p \in (1, \infty)$, and in fact,

$$\|\mathcal{K}(f)\|_p \leq C \|f\|_p .$$

(b) *Schur's Lemma:* This is a souped-up version of Young's Test, which lets us deal with more difficult kernels.
Let $\Omega \subseteq \mathbb{C}^n$ be a domain, K be an a.e. positive, measurable function on $\Omega \times \Omega$, and $\mathcal{K}$ be the integral operator with kernel K. Suppose there exists a positive auxiliary function h on Ω, and a number $a > 0$ such that for all $\epsilon \in [0, a)$, the following estimates hold:

$$\mathcal{K}(h^{-\epsilon})(z) := \int_\Omega K(z,w) h(w)^{-\epsilon}\, dV(w) \lesssim h(z)^{-\epsilon}$$
$$\mathcal{K}(h^{-\epsilon})(w) := \int_\Omega K(z,w) h(z)^{-\epsilon}\, dV(z) \lesssim h(w)^{-\epsilon}.$$

Then $\mathcal{K}$ is a bounded operator on $L^p(\Omega)$, for all $p \in (1, \infty)$.

(c) Show that the hypotheses of Schur's Lemma are satisfied when the domain $\Omega = \mathbb{D}$, the kernel $K(z,w) = |B_{\mathbb{D}}(z,w)|$, and the auxiliary function $h(z) = 1 - |z|^2$.

Note: *In part (c), h(z) is (essentially) the distance to the boundary function. In practice, this is usually a good choice for an auxiliary function. Using similar ideas coupled with more sophisticated machinery, it can be shown that the Bergman projection is an* L^p *bounded operator for* $1 < p < \infty$ *on a large class of smoothly bounded, pseudoconvex domains. For example, this is known to hold for all smooth bounded,* **strongly** *pseudoconvex domains. However, we will see later that such a result fails in general.*