EXERCISE SET 1

Lecture 1

Exercise 1. Given an exact sequence $0 \to A \to B \to C \to 0$, prove that the complex
$$[A \to B \to C]$$
is homotopy equivalent to 0 iff the sequence splits.

Exercise 2. Give an example of a complex C with nontrivial cohomology only in degree zero, but with:
(1) no chain maps $H^0(C) \to C$;
(2) no chain maps $C \to H^0(C)$;
(3) no chain maps either way.

Exercise 3. Construct a morphism
$$[\mathbb{Z} \xrightarrow{2} \mathbb{Z}] \to [\mathbb{Z} \to \mathbb{Z}/3]$$
inducing zero on cohomology, which is nonzero in derived category.

Exercise 4. Which complexes become zero in derived category? Give an example of a complex, zero in derived category, but not zero in homotopy category.

Exercise 5. An abelian category is called *semisimple* if every short exact sequence splits. Prove that the derived category of a semisimple abelian category is equivalent to the category of graded objects, i.e. complexes with zero differential.

Exercise 6. Let $\mathcal{C}$ be a full abelian subcategory in an abelian category $\mathcal{A}$, closed under passing to subobjects and quotients and closed under extensions. Let S be the class of morphisms consisting of all f in $\mathcal{A}$ such that $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{C}$. Prove that S is a localizing class.

Exercise 7. Show that quasi-isomorphisms do not form a localizing class in the category of complexes of abelian groups (rather than the homotopy category). Precisely, show that you cannot fill in the diagram
$$\begin{tikzcd} ? \arrow[r, dotted, "\text{qiso}"'] \arrow[d, dotted] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "\text{qiso}"'] & B^\bullet \end{tikzcd}$$
if $A^\bullet = 0$, $B^\bullet = [\mathbb{Z} \to \mathbb{Z}]$, $C^\bullet = [0 \to \mathbb{Z}]$, and f and g are the obvious maps.

Lecture 2

Exercise 8. Let $f\colon A^\bullet \to B^\bullet$ be a chain map, and let $\tau\colon B^\bullet \to \operatorname{cone} f$ be the natural map. Show that the pair of maps $A^\bullet[1] \longleftrightarrow \operatorname{cone}\tau$ defined in lecture are homotopy inverse to one another.

Exercise 9. Show that the cone of $\tau_{\le 0}C^\bullet \to C^\bullet$ is quasi-isomorphic to $\tau_{\ge 1}C^\bullet$.

Exercise 10. The Koszul complex, $K(a_1, ..., a_n)$, associated to $a_1, \ldots, a_n \in R$ can be viewed as the cone of the multiplication by a_n morphism:

$$K(a_1, \ldots, a_n) \cong \mathrm{Cone}(a_n \colon K(a_1, ..., a_{n-1}) \to K(a_1, ..., a_{n-1}))$$

Use this and induction to prove that the Koszul complex of a regular sequence is exact (except in the last place). *Hint: In the induction step we can replace $K(a_1, ..., a_{n-1})$ by $R/(a_1, ..., a_{n-1})$.*

Exercise 11. Let $R = k[x, y]$, and let k also denote the R-module $R/(x, y)$. (The corresponding sheaf on $\mathbb{A}^2$ is the skyscraper sheaf at the origin.) Interpret a non-trivial element of $\mathrm{Ext}^2_R(k, k)$ as a chain map from a complex of free modules that's quasi-isomorphic to k, to the complex $k[2]$.

Exercise 12. On $\mathbb{P}^1$, interpret a non-trivial element of $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-2))$ as a chain map from a complex that's quasi-isomorphic to $\mathcal{O}$, to the complex $\mathcal{O}(-2)[1]$.

Exercise 13. Give an example of a bounded complex in an abelian category which is not "formal", i.e. not isomorphic in derived category to a complex with zero differentials.

Exercise 14. Show that the Yoneda description of Ext^n as equivalence classes of n-step extensions is equivalent to the usual definition by taking injective or projective resolutions.

Lecture 3

Exercise 15. Use the general criterion from the lecture to prove that $\mathcal{D}^-$ (resp., $\mathcal{D}^+$, $\mathcal{D}^b$) is a full subcategory of $\mathcal{D}$.

Exercise 16. Give an example of an acyclic unbounded complex of projective modules, which is not homotopy equivalent to zero.