High dimensions are weird
Week 2 Exercise problems

Write B_k^n for the n-dimensional L_k unit ball: $B_k^n := \{x : x \in \mathbb{R}^n, \|x\|_k \leq 1\}$. Similiarly, denote S_k^n as the analogous sphere: $S_k^n := \{x : x \in \mathbb{R}^n, \|x\|_k = 1\}$. Note these geometric objects have unit radius. Define the unit-sided cube by $C^n := \frac{1}{2} B_\infty^n$.

(1) For $n = 2$, compute the volume (area) of B_k^n for $k = 1, 2, 3$. Also compute the volume of C^n for any n.

(2) **Challenging calculus flashback.** Compute the volume of the n-dimensional sphere B_2^n. If you want to check/view the result, you can go here: `https://en.wikipedia.org/wiki/Volume_of_an_n-ball`.

(3) Use your answer from above to show that as n grows large, the volume of the n-dimensional sphere goes to zero.

(4) For $n = 2$, show that the distance from the origin to any vertex of C^2 is $\sqrt{2}/2$. Use this to argue that the cube lies completely inside the unit circle.

(5) Show that for $n = 4$ the distance from the origin to any vertex of C^4 is 1, and thus the cube "barely" fits inside the unit ball B_2^4.

(6) Generalize the above to show that the distance from the origin to a vertex of the unit cube scales like $\sqrt{n}/2$, and so for large n, the vertices lie far away from the unit sphere.

(7) Show that on the other hand, the *midpoint* of any face of the unit cube lies a distance only 1/2 from the origin. Therefore, the "middle" of each face lies well within the unit sphere.

(8) Use your answers to the above to "draw" a schematic for how the unit cube and unit sphere relate in high dimensions. Use these answers to argue that "most" of the volume of the cube is located outside of the sphere.

(9) Try to prove this more formally. In other words, show that for any $c > 0$, the fraction of the volume of the hemisphere above the plane $x_1 = c/\sqrt{n-1}$ is less than $\frac{2}{c} e^{-c^2/2}$. If you get stuck, there are some nice notes here: `https://www.cs.cmu.edu/~venkatg/teaching/CStheory-infoage/chap1-high-dim-space.pdf` (see Lemma 1.2).