Dynamics of Rational Skew Products

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Algebraic Stability

f is *algebraically stable (AS)* iff $\forall n\ (f^*)^n = (f^n)^*$, for $f^* : H^*(X) \to H^*(X)$ or on $\mathrm{Pic}(X)$.
On a surface X, this corresponds to f **not** having any *destabilising orbits*:
A sequence of points $p, f(p), f^2(p), \ldots, f^n(p)$ such that $f^{-1}(p)$ and $f^{n+1}(p)$ are curves.
Fact: When f is algebraically stable, then its first (and other) dynamical degree(s), $\lambda = \lim \|(f^n)^*\|^{\frac{1}{n}}$ are algebraic integers.

Rational Skew Products

A *rational skew product* $f : X \dashrightarrow X$ (on a surface X) is a rational map respecting a fibration $h : X \to B$ over a curve B.

More intuitively, this is a map which is locally given by

$$(x, y) \longmapsto (f_1(x), f_2(x, y)).$$

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ B & \underset{f_1}{\longrightarrow} & B \end{array}$$

Questions

- Which f are algebraically stable?
- If a rational map f is not algebraically stable is there a birational conjugacy given by $\phi : Y \dashrightarrow X$ which *stabilises* f?
  i.e. $g = \phi^{-1} \circ f \circ \phi$ is algebraically stable.

$$\begin{array}{ccc} Y & \overset{g}{\dashrightarrow} & Y \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X & \underset{f}{\dashrightarrow} & X \end{array}$$

Known Results

Any continuous f is algebraically stable.

- Diller-Favre ('01): When $f : X \dashrightarrow X$ is birational, f can be stabilised by a birational morphism (sequence on blowups) $\pi : \hat{X} \to X$.
- Favre ('03): Many monomial maps on $\mathbb{P}^2$ cannot be stabilised by any birational ϕ.
- Diller-Lin ('16): A certain class of monomial-like maps cannot be stabilised.

Our Objectives

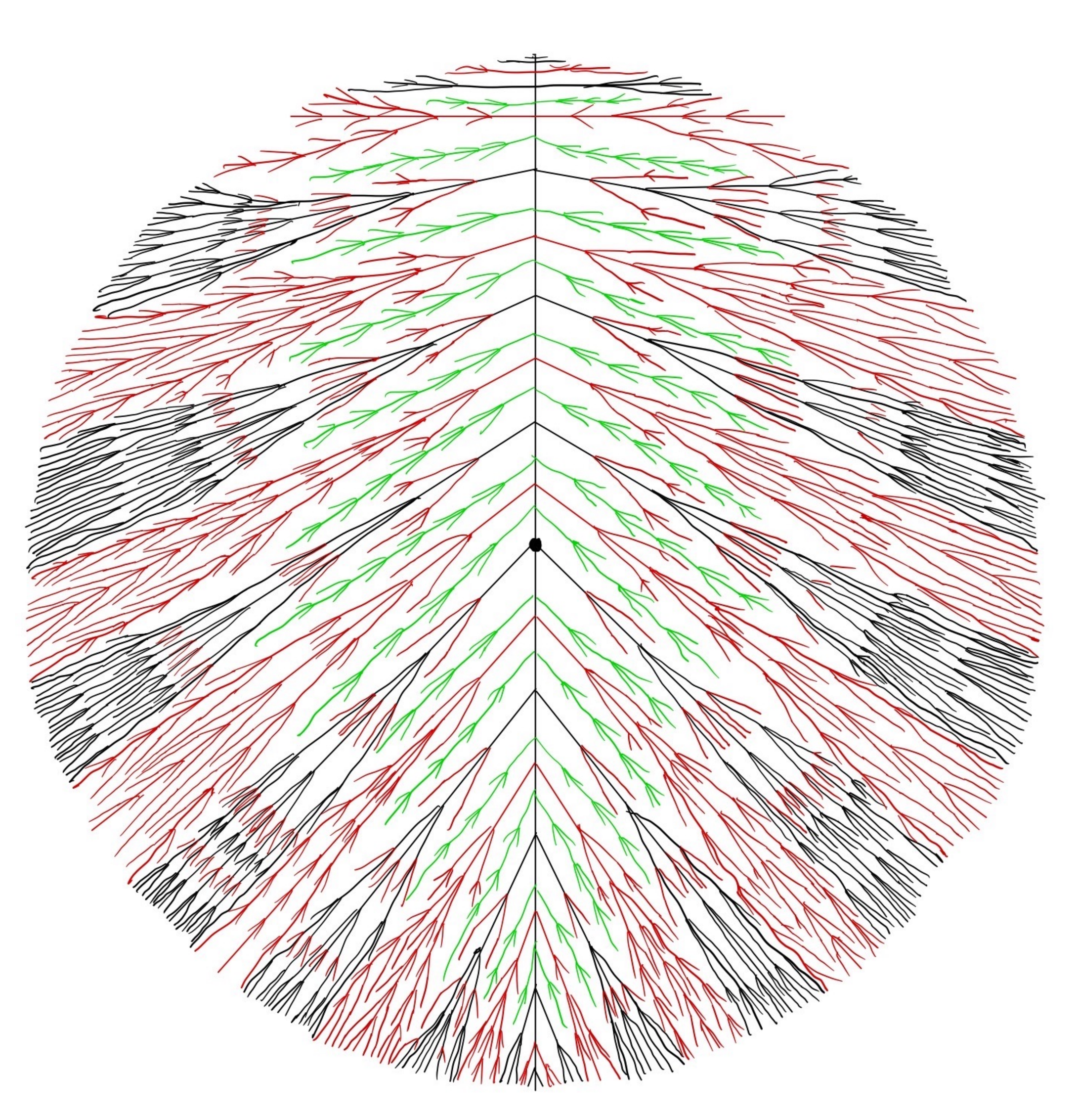
Rational skew products provide a rich context for investigating algebraic stability because despite being rational, f has to send curves of one fibre to curves of another fibre.
Intriguingly, they also have integer first dynamical degree $\lambda_1 = \max\{\deg(f_1), \mathrm{rdeg}(f)\}$ where the *relative degree* $\mathrm{rdeg}(f)$ is the y-degree of $f_2 \in \mathbb{C}(x)(y)$. This suggests that skew products are algebraically stable, if you believe the converse of the fact above. Unfortunately this converse has already been shown false as shown by Favre for monomial maps. Regardless, maybe this is a fault of monomial maps; can skew products be stabilised?

Counter-Example

Theorem 1 (B.) *Consider the map on $\mathbb{P}^1 \times \mathbb{P}^1$ given by*

$$f : (x, y) \longmapsto \left((1 - x)x^2, (1 - x)(x^4 y^{-3} + y^3)\right).$$

Then f is not algebraically stable after any *birational conjugation.*



The Berkovich Projective Line

The *Berkovich affine line* $\mathbb{A}^1_{\mathrm{an}}(L)$ is the set of multiplicative seminorms $\zeta = \|\cdot\|_\zeta$ on $L(y)$ which extend a fixed non-Archimedean norm $|\cdot|$ on L. The *Berkovich projective line* $\mathbb{P}^1_{\mathrm{an}}(L)$ compactifies this with a point ∞.
Let $L = \overline{K}$, where

$$K = \widehat{\mathbb{C}((x))} = \left\{ \sum_j c_j x^{\frac{r_j}{s_j}} : c_j \in \mathbb{C}, s_j \text{ bounded} \right\}$$

is the field of *Puiseux series*. We give it the non-Archimedean norm $|\cdot|$ defined by $|\mathbb{C}^*| = 1$ and $0 < |x| < 1$.

Given a rational skew product f, we have a natural extension from $f^* : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$ to a map $f^* : L(y) \to L(y)$. We normally define a map $f_* : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{an}}$ on the Berkovich line by $\|\phi\|_{f_*(\zeta)} = \|f^*(\phi)\|_\zeta$. However every seminorm must extend the norm on L, $f_*(\zeta)$ included. In particular we need that $\|x\|_{f_*(\zeta)} = |x|$.
Unlike in the case of a rational map on $\mathbb{P}^1_{\mathrm{an}}$ where $f_1 = \mathrm{id}$, we have

$$f_1(x) = c_d x^d + c_{d+1} x^{d+1} + \cdots.$$

This means that $\|x\|_{f_*(\zeta)} = \|f^*(x)\|_\zeta = \|f_1(x)\|_\zeta = |x|^d$. By renormalising to $\|f^*(\cdot)\|_\zeta^{\frac{1}{d}}$, we get a well-defined continuous map f_*. This is similar to the higher dimensional case in the valuative tree, see Favre-Jonsson.

Fatou - Julia Theory

Let f be a simple skew product. $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ is *Fatou* iff it has a neighbourhood $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ such that $\bigcup_{n \geqslant 0} f^n(U)$ omits infinitely many points. Denote the *Fatou set* by $\mathcal{F}_{f,\mathrm{an}}$ and the *Julia set* by $\mathcal{J}_{f,\mathrm{an}} = \mathbb{P}^1_{\mathrm{an}} \setminus \mathcal{F}_{f,\mathrm{an}}$.
Fortunately, all the Fatou-Julia theory for 'rational maps' on the Berkovich line carry over to rational skew products. In particular we generalise the *Classification of Fatou Components*, by Rivera–Letelier.

Theorem 2 (B., Rivera-Letelier) *Let f be a simple rational skew product of relative degree $\mathrm{rdeg}\, f \geqslant 2$ with a Berkovich Fatou component $U \subseteq \mathcal{F}_{f,an}$. Then either (i) U is a wandering domain (ii) $\exists n$ such that $f_*^n(U)$ is an indifferent component, or (iii) $\exists n$ such that $f_*^n(U)$ is an attracting component.*

Stabilising Skew Products

Inspired by the theorem of DeMarco-Faber ('15) in the case where $f_1 = \mathrm{id}$, we get a positive result for skew products on surfaces where the generic fibre is rational.

Theorem 3 (B.) *Let $h : X \to B$ be a ruled surface and $f : X \dashrightarrow X$ be a rational skew product such that $f_1 : B \to B$ has no superattracting points. Then there is a birational morphism $\pi : \hat{X} \to X$, blowing up X, such that the conjugate $\hat{f} : \hat{X} \dashrightarrow \hat{X}$ is algebraically stable.*