MSRI Summer School: Incompressible Fluid Flows at High Reynolds Number
Howmework 1
July 27, 2015

All references to exercise, chapter, or appendix numbers to the *lecture notes* made available by the instructors. A *minimal* dosage of exercises to solve may be measured as follows:

- gather 3* from problems for *each* of the following in Sections: 1, 2, 5, 7;
- gather 7* from problems in Section 3;
- gather 1* from the problem in Section 4;
- gather 12* from problems in Section 6.

1. Ordinary differential equations

EXERCISE 1.1 (**). *Prove Theorem A.5 (Peano existence theorem) from Appendix A of the lecture notes.*

EXERCISE 1.2 (*). *Do Exercise A.3 from Appendix A of the lecture notes.*

EXERCISE 1.3 (***). *Prove Theorem A.4 (Osgood uniqueness condition) from Appendix A of the lecture notes.*

2. Functional Analysis

Recall the following classical result (which follows from the Baire category theorem):

THEOREM (Banach–Steinhaus, aka Uniform boundedness principle). Let X and Y be two Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $\{T_i\}_{i\in I}$ be a family of continuous linear operators from X into Y, i.e. $T_i \in L(X,Y)$. Assume that

$$\sup_{i\in I} \|T_i x\|_Y < \infty \qquad \text{for all } x \in X. \tag{2.1}$$

Then we have

$$\sup_{i\in I} \|T_i\|_{L(X,Y)} < \infty \tag{2.2}$$

where

$$\|T\|_{L(X,Y)} = \sup_{x\in X:\ \|x\|_X \le 1} \|Tx\|_Y \tag{2.3}$$

is the norm on the bounded linear operators from X to Y.

EXERCISE 2.1 (**). *Let X be a Banach space with norm $\|\cdot\|$. Consider $x_n \rightharpoonup x$ as $n \to \infty$, a weakly convergent sequence (recall this means that given any $f \in X^*$ we have $f(x_n) \to f(x)$). Show that $\{x_n\}$ is bounded.*
Hint. *Use the Uniform Boundedness Principle. Given a functional $f \in X^*$, it may be useful to consider the linear operators $T_n(f) := f(x_n)$, which are elements of X^{**}.*

EXERCISE 2.2 (***). *Let X be a Banach space with norm $\|\cdot\|$. Consider $x_n \rightharpoonup x$ as $n \to \infty$, a weakly convergent sequence. Show that $\|x\| \le \liminf_{n\to\infty} \|x_n\|$.*
Hint. *Here you may use without proof a consequence of the Hahn-Banach theorem: there exists functional $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.*

EXERCISE 2.3 (*). *Give an example of $f_n \in L^2([0, 2\pi])$ with $\|f_n\|_{L^2} = 1$ such that $f_n \rightharpoonup 0$ as $n \to \infty$.*

EXERCISE 2.4 (*). *Let X be a Hilbert space with inner product $(\cdot,\cdot)$. Prove that if $x_n \rightharpoonup x$ as $n \to \infty$ and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then $x_n \to x$ strongly.*

3. Sobolev spaces

EXERCISE 3.1 (*). *Show that $H^1(\mathbb{T}^2) \not\subset L^\infty(\mathbb{T}^2)$, i.e. do Exercise B.1 in Appendix B of the lecture notes.*

EXERCISE 3.2 (**). *Prove that the endpoint Sobolev embedding is not locally compact, i.e. do Exercise C.3 in Appendix C of the lecture notes.*

EXERCISE 3.3 (**). *Prove that Rellich's theorem fails on unbounded domains, i.e. do Exercise C.5 in Appendix C of the lecture notes.*

EXERCISE 3.4 (**). *Do Exercise C.4 in Appendix C of the lecture notes.*

EXERCISE 3.5 (*). *Prove the interpolation inequality of Exercise C.7 in Appendix C.*
Hint. *By contradiction; use that since X is reflexive the unit ball is weakly compact (Banach-Alaoglu).*

EXERCISE 3.6 (**). *Prove the following version of the Gagliardo-Nirenberg inequality. Let $f \in C_0^\infty(\mathbb{R}^d)$, let $m \geq 2$ be an integer, and let $i \in \{0, 1, \dots, m\}$. Prove that there exists $C = C(d, i, m) > 0$ such that*

$$\|\partial^\alpha f\|_{L^{2m/i}} \leq C \|f\|_{L^\infty}^{1-\frac{i}{m}} \|f\|_{\dot{H}^m}^{\frac{i}{m}} \tag{3.1}$$

for any multi-index α with $|\alpha| = i$.
Hint. *Do it first for $m = 2$ and $i = 1$, where you simply need to integrate by parts. Now do it for $m = 3$ and $i = 1$. See the pattern?*

EXERCISE 3.7 (**). *Prove Proposition G.8 (Product Rule for Sobolev Spaces) from Appendix G of the lecture notes.*

EXERCISE 3.8 (**). *Use Littlewood-Paley to prove yet another kind of Gagliardo-Nirenberg-Sobolev-type inequality: For all $1 < p < q \leq \infty$ and $s > 0$, the following holds for all $u \in \mathcal{S}(\mathbb{R}^d)$ (and by approximation, any function for which the RHS is finite*

$$\|u\|_{L^q} \lesssim_{p,q,d,s} \|u\|_{L^p}^{1-\theta} \left\| |\nabla|^s u \right\|_{L^p}^{\theta},$$

if the following relation is satisfied

$$\frac{d}{q} = \frac{d}{p} - \theta s.$$

4. Mollifiers

EXERCISE 4.1 (*). *Prove Proposition D.5 from Appendix D of the lecture notes.*

5. Singular integrals and continous functions

EXERCISE 5.1 (***). *A function $\rho\colon [0,\infty) \to [0,\infty)$ is called a modulus of continuity if ρ is non-decreasing, continuous, concave and $\rho(0) = 0$. We say that the function $f\colon \mathbb{R}^d \to \mathbb{R}$ obeys the modulus of continuity ρ if $|f(x) - f(y)| \leq \rho(|x-y|)$ for all $x, y \in \mathbb{R}^d$. Consider T a Calderon-Zygmund operator of convolution type with*

$$Tf(x) = P.V. \int_{\mathbb{R}^d} \frac{\Omega(\widehat{y})}{|y|^d} f(x-y)\,dy \tag{5.1}$$

where Ω is smooth on $\mathbb{S}^{d-1}$, and has zero mean on the unit sphere. Show that if f obeys the modulus of continuity ρ, then the function Tf obeys the modulus of continuity

$$\widetilde{\rho}(r) = C \int_0^r \frac{\rho(t)}{t}\,dt + Cr \int_r^\infty \frac{\rho(t)}{t^2}\,dt \tag{5.2}$$

for some universal constant $C > 0$. As a corollary, show that $T\colon L^2 \cap C^\alpha \to C^\alpha$ for $\alpha \in (0,1)$.

6. Euler equations

EXERCISE 6.1 (**). *Show that the divergence of u stretches the Jacobian of the Lagrangian map associated to u, i.e. do Exercise 1.2 in Section 1 of the lecture notes.*

EXERCISE 6.2 (**). *Prove that the pressure gradient may be bounded quadratically in terms of the velocity in suitable Sobolev spaces, i.e. do Exercise 1.11 in Section 1 of the lecture notes.*

EXERCISE 6.3 (****). *Obtain a formula for the Euler pressure on the half space, i.e. do Exercise 1.12 in Section 1 of the lecture notes.*

EXERCISE 6.4 (*). *Do Exercise 1.14 in Section 1 of the lecture notes.*

EXERCISE 6.5 (***). *Obtain a formula for the stretching of the modulus of vorticity in 3D Euler, i.e. do Exercise 1.20 in Section 1 of the lecture notes.*

EXERCISE 6.6 (*). *Do Exercise 1.22 from Section 1 of the lecture notes.*
Hint. *Use Exercise 5.1 from the Singular Integrals section of this homework sheet.*

EXERCISE 6.7 (**). *Consider a solution u of the 2D Euler equations such that $u \in C([0,T]; L^2(\mathbb{R}^2))$ and $\omega \in C([0,T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$. Show that for all $t \in [0,T]$ there* **exists** *a* **unique** *particle trajectory $X(a,t)$ associated to each point $a \in \mathbb{R}^2$, i.e. the solution of*

$$\frac{dX(a,t)}{dt} = u(X(a,t),t), \qquad X(a,0) = a, \tag{6.1}$$

exists and is unique.
Hint. *Think of the existence and uniqueness theorems for ODEs. Note that u will not be Lipschitz, since the map $\omega \mapsto \nabla u$ does not map L^∞ to L^∞ (but it doesn't miss by much). Look at Corollary F.6 from Appendix F of the lecture notes, and combine with the Osgood criterion.*

EXERCISE 6.8 (***). *Derive the evolution of a vortex patch in 2D Euler. That is, solve Exercises 1.23 and 1.24 in Section 1 of the lecture notes.*

EXERCISE 6.9 (*). *Prove that helicity is conserved in 3D Euler, i.e. do Exercise 1.35 in Section 1 of the lecture notes.*

EXERCISE 6.10 (**). *Prove that 3D Euler is not locally well posed in $C^\alpha(\mathbb{T}^3)$, when $\alpha \in (0,1)$, i.e. do Exercise 1.36 in Section 1 of the lecture notes.*

7. Navier-Stokes

EXERCISE 7.1 (**). *Complete the proof of d'Alambert's paradox, that is do Exercises 2.2 and 2.3 in Section 2 of the lecture notes.*

EXERCISE 7.2 (*). *Prove the exponential decay of energy in 2D unforced NSE on the torus under the mean free condition, i.e. do Exercise 2.5 in Section 2 of the lecture notes. What happens as $t \to \infty$ if we add a time independent $f \in L^2(\mathbb{T}^2)$ on the RHS of the equations?*

EXERCISE 7.3 (*). *Prove the decay of enstrophy in 2D unforced NSE on the plane, i.e. do Exercise 2.5 in Section 2 of the lecture notes.*