1 The Euler equations

Our goal is to study the motion of fluids at a macroscopic level: instead of keeping track of the movement on every molecule, we consider some "average behavior" or "mean field approximation". For this purpose we regard the fluid as a continuum, and in this description we have in mind that a *fluid particle/ point in the fluid/ infinitesimal volume* is: large enough to contain a large number of molecules, and much larger than the distance between "neighboring" molecules; and small enough (infinitesimal) with respect to the size of the domain over which the motion is considered. This approximation makes sense if the distances over which the quantities considered vary significantly is much larger than the atomic scale.

1.1 Mathematical description of fluid flow

The *Eulerian* quantities we use to describe an ideal incompressible fluid occupying a domain

$$\Omega \subset \mathbb{R}^d$$

where $d \in \{2, 3\}$ is the spatial dimension, are

- the *velocity* vector field (as experienced at a fixed location in the fluid)

  $$u(x,t) = (u_1, \ldots, u_d)(x,t) \colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$$

  with units $[u] = LT^{-1}$;

- the *pressure* scalar field (force acting on a fluid particle due to the surrounding fluid particles)

  $$p(x,t) \colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$$

  with units $[p] = ML^{-1}T^{-2}$;

- the *density* scalar field (mass per unit volume in the fluid)

  $$\rho(x,t) \colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$$

  with units $[\rho] = ML^{-3}$;

where $x \in \mathbb{R}^d$ and $t \in [0,\infty)$. Here we use $[x] = L$, $[t] = T$, and M as some fixed units of measurement for space, time and mass. For example $L = 1$ meter, $T = 1$ second, and $M = 1$ kilogram. We emphasize that in the Eulerian description the unknowns are measured at a stationary position (x,t) in space time.

Instead, we may wish to consider the change in a quantity as experienced by a particle that is travelling withe the fluid. This is the *Lagrangian* description. Given an initial configuration of particles, labeled by $a \in \Omega$, the unknowns are the *particle trajectories* at times $t > 0$:

$$X(a,t) = (X_1, \ldots, X_d)(a,t) \colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d.$$

The map

$$X(\cdot,t) \colon \Omega \to \Omega, \qquad a \mapsto X(a,t)$$

is called the *flow map*. We will later show that under suitable conditions this map is an (volume-preserving) isomorphism of Ω. Assuming for the moment that this is true, we denote the inverse of X by

$$A(x,t) = (A_1, \ldots, A_d)(x,t) \colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$$

which obeys

$$A(X(a,t),t) = a, \qquad X(A(x,t),t) = x$$

for all $a, x \in \mathbb{R}^d$. The map A is called the *back-to-labels* map, and we sometimes write $A = X^{-1}$.

The connection between the Eulerian and the Lagrangian description of fluid flow is given by the fact that the particle trajectories X move on the integral curves of the velocity field u, that is they obey the ODE

$$\partial_t X(a,t) = u(X(t,a),t) \tag{1.1}$$

with initial conditions

$$X(a,0) = a. \tag{1.2}$$

Geometrically, this means that the velocity field u, evaluated at position $(x,t) = (X(a,t),t)$, is tangent to the curve described by the motion of the particle a, namely $\{X(a,\tau)\}_{\tau\in I}$, where $t \in I$, at the point $X(a,t)$.

1.2 Convective derivative

Let f be a function $f(x,t)\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ and consider some fixed point $x \in \mathbb{R}^3$. The instantaneous rate of change of f with respect to time t, at the fixed position x, is classically given by the partial time derivative ∂_t as

$$\partial_t f(x,t) = \lim_{\Delta t\to 0} \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}.$$

This rate of change is *static*: it is experienced by a fixed observer in spaces.

On the other hand let us consider a *dynamic* rate of change: as experienced by an observer moving with the fluid along the particle trajectories X determined by a velocity field u. This defines the *convective derivative* D_t by

$$\begin{aligned} D_t f(x,t) &= \lim_{\Delta t\to 0} \frac{f(X(x,\Delta t),t+\Delta t) - f(x,t)}{\Delta t} \\ &= \lim_{\Delta t\to 0} \frac{f(x+u(x,t)\Delta t,t+\Delta t) - f(x,t)}{\Delta t} \\ &= \partial_t f(x,t) + \lim_{\Delta t\to 0} \frac{f(x+u(x,t)\Delta t,t+\Delta t) - f(x,t+\Delta t)}{\Delta t} \\ &= \partial_t f(x,t) + \sum_{i=1}^{d} \partial_{x_i} f(x,t) u_i(x,t) \\ &= (\partial_t + u\cdot\nabla) f(x,t). \end{aligned} \tag{1.3}$$

Here we have used that f and u are C^1 smooth functions, so that for instance $X(x,\Delta t)$ is given from (1.1)–(1.2) to leading order in Δt by $x + u(x,t)\Delta t + O((\Delta t)^2)$. Upon using the chain rule, formula (1.3) becomes

$$\partial_t\left(f(X(a,t),t)\right) = (D_t f)(X(a,t),t) \tag{1.4}$$

which is precisely measuring the change in time of f as experienced by a particle moving along the integral curves of u.

We note that from this definition, we have that the acceleration of a fluid particle, i.e. the change in velocity as experienced by a particle moving with the fluid of velocity u, is given by

$$\begin{aligned}\partial_{tt}X(a,t) &= \partial_t\left(u(X(a,t),t)\right)\\ &= (D_t u)(X(a,t),t)\\ &= (\partial_t u + (u\cdot\nabla)u)(X(a,t),t).\end{aligned}$$

Here we have again appealed to (1.1).

1.3 On the existence and uniqueness of the Lagrangian particle trajectories

For every fixed label a, the system (1.1)–(1.2) is an ordinary differential equation with respect to t. In view of the classical Peano existence theorem (see Theorem A.5), this ODE has at least one solution if $u \in C_t^0 C_x^0$. Moreover, the Picard-Lindelöf theorem (see Theorem A.2) asserts that there exists at precisely one solution if in addition $u \in C_t^0 Lip_x$. This uniqueness condition is almost sharp, see the Osgood condition in Theorem A.4. Throughout these notes we shall assume that we are working on a time interval on which u is sufficiently smooth to ensure that (1.1)–(1.2) has a unique solution. We next wish to show that under suitable smoothness condition on u, the flow maps X, and are inevitable diffeomorphisms. Assuming that $u \in C_t^0 Lip_x$, the gradient with respect to the label a of the path $X(a,t)$ obeys

$$\partial_t \frac{\partial X_j}{\partial a_k}(a,t) = \frac{\partial u_j}{\partial x_i}(X(a,t),t)\frac{\partial X_i}{\partial a_k}(a,t), \qquad \frac{\partial X_j}{\partial a_k}(a,0) = \delta_{jk}$$

where δ_{jk} is the Kronecker symbol (i.e. the (j,k) entry of the identity matrix). From the Gönwall inequality we then obtain

$$\sup_{a\in\mathbb{R}^d} |\nabla_a X(a,t)| \leq \exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty} ds\right). \tag{1.5}$$

Recall that by definition the *back-to-labels* map $A(x,t) = X^{-1}(x,t)$ obeys $x = X(A(x,t),t)$. Differentiating with respect to time the last identity, using the chain rule and (1.1), we obtain that

$$\frac{\partial X_k}{\partial a_j}(A(x,t),t)\partial_t A_j(x,t) = -\partial_t X_k(A(x,t),t) = -u_k(x,t). \tag{1.6}$$

On the other hand, differentiating with respect to a the identity $a = A(X(a,t),t)$, we obtain

$$\delta_{ij} = \frac{\partial A_i}{\partial x_k}(X(a,t),t)\frac{\partial X_k}{\partial a_j}(a,t)$$

Therefore, multiplying (1.6) from the left with the matrix $\partial_x A$, we obtain the evolution equation obeyed by the back to labels map

$$\partial_t A + u\cdot\nabla A = 0 \tag{1.7}$$

pointwise for (x,t) in the domain, with initial datum $A(x,0) = x$. Applying a gradient with respect to x to (1.7) we arrive at

$$\partial_t(\partial_i A_j) + u_k \partial_k(\partial_i A_j) + (\partial_i u_k)(\partial_k A_j) = 0, \qquad \partial_i A_j(x,0) = \delta_{ij},$$

where the partial derivatives ∂_i are derivatives with respect x_i. Using (1.4), we may write the above after evaluating at $x = X(a,t)$ as

$$\partial_t(\nabla_x A)(X(a,t),t) = -(\nabla_x u)(X(a,t),t)(\nabla_x A)(X(a,t),t)$$

where on the right side of the above we have a matrix product. From the Grönwall inequality (in matrix form), and the fact that at time $t = 0$, we have $\nabla_x A = \mathrm{Id}$, we obtain that

$$\sup_{a\in\mathbb{R}^d} |\nabla_x A(X(a,t),t)| \leq \exp\left(\int_0^t \|\nabla u(s)|_{L^\infty} ds\right). \tag{1.8}$$

Since

$$a - b = A(X(a,t),t) - A(X(b,t),t)$$

it follows from the mean value theorem and (1.8) that

$$|a-b| \leq |X(a,t) - X(b,t)| \exp\left(\int_0^t \|\nabla u(s)|_{L^\infty} ds\right).$$

One the other hand, the mean value theorem combined with (1.5) yields

$$|X(a,t) - X(b,t)| \leq |a-b| \left(\int_0^t \|\nabla u(s)|_{L^\infty} ds\right).$$

Combining the above two inequalities we have thus proven the chord-arc condition

$$\left(-\int_0^t \|\nabla u(s)|_{L^\infty} ds\right) \leq \frac{|a-b|}{|X(a,t) - X(b,t)|} \leq \left(\int_0^t \|\nabla u(s)|_{L^\infty} ds\right) \tag{1.9}$$

for all $a, b \in \Omega$, and all $t \geq 0$ as long as we have $u \in L^1_t Lip_x$ up to time t. The upshot of (1.9) is that it ensures the invertibility of the Lagrangian map X, and shows that both X and A are Lipschitz continuous.

1.4 Conservation of mass

Let V be a volume element in the fluid. The density ρ determines the mass of this volume element by

$$m(V,t) = \int_V \rho(x,t)dx.$$

We assume that *mass is neither created nor destroyed* in a volume element moving with the fluid, i.e. that

$$\frac{d}{dt} m(V(t),t) = 0. \tag{1.10}$$

Assuming that ρ, u, and ∂V are sufficiently smooth, and using that the mass flow per unit area crossing the boundary of the fluid element V is given by $\rho u \cdot n$, where $n = n(x)$ is the outward unit normal to ∂V. In Eulerian coordinates we may thus write

$$\begin{aligned}
\int_V \partial_t \rho(x,t)dx &= \frac{d}{dt}\left(\int_V \rho(x,t)dx\right) \\
&= -\int_V \rho(x,t)u(x,t)\cdot n(x)dx \\
&= -\int_V \nabla\cdot(\rho(x,t)u(x,t))dx
\end{aligned}$$

where V is a fixed volume element. In the last line of the above we have used the divergence theorem. The conservation of mass assumption (1.10) is thus equivalent to the fact that

$$\int_V \left(\partial_t \rho + \nabla\cdot(\rho u)\right)(x,t)dx = 0 \tag{1.11}$$

for any smooth volume element $V \subset \Omega$. Another way to derive the above is to use Lemma 1.4 below (with $f = \rho$), and show that (1.11) is a direct consequence of (1.10), i.e.

$$0 = \frac{d}{dt} m(V(t), t) = \frac{d}{dt} \left(\int_{V(t)} \rho(x, t) dx \right) = \int_{V(t)} (\partial_t \rho + \nabla \cdot (\rho u)) (x, t) dx.$$

If ρ and u are smooth, the integral equality (1.11) yields the *conservation of mass equation*

$$\partial_t \rho + \nabla \cdot (\rho u) = 0 \tag{1.12}$$

pointwise at $(x, t) \in \Omega \times \mathbb{R}$.

1.5 Incompressibility

Let V be a volume element in the fluid, and denote by

$$V(t) = X(V, t) = \{X(a, t) \colon a \in V\}$$

the pushfoward of this set under the flow map $X(\cdot, t)$ determined by the velocity field u. We say that the velocity field u is *incompressible* if the flow map $X(\cdot, t)$ is *volume-preserving*, meaning that

$$|V| = |V(t)| \tag{1.13}$$

for all $V \subset \Omega$, and all $t \in \mathbb{R}$. Here and throughout we denote by $|A|$ the Lebesgue measure of a set A. As it turns out, assumption (1.13) is equivalent to

$$\nabla \cdot u = 0 \tag{1.14}$$

namely that the velocity field is *divergence free*. In order to prove the equivalence of (1.13) and (1.14) we recall the change of variables formula

$$\begin{aligned} \int_{V(t)} f(x) dx &= \int_V f(X(a, t)) \det(\nabla_a X)(a, t) da \\ &= \int_V f(X(a, t)) J(a, t) da \end{aligned}$$

where we have denoted by $J(a, t)$ the determinant of Jacobian $\nabla_a X$ associated to the map $a \mapsto X(a, t)$, i.e.

$$J(a, t) = \det(\nabla_a X)(a, t) = \sum_{i_1, \ldots, i_d = 1}^{d} \varepsilon_{i_1 \ldots i_d} \frac{\partial X_{i_1}}{\partial a_1}(a, t) \ldots \frac{\partial X_{i_d}}{\partial a_d}(a, t) \tag{1.15}$$

where $\varepsilon_{i_1 \ldots i_d}$ denotes the standard Levi-Civita symbol (recall, this equals the signum $\operatorname{sgn}(\sigma)$ if $i_j = \sigma(j)$ for all $j \in \{1, \ldots, d\}$ and some permutaiton $\sigma \in S_d$, and it equals 0 otherwise). One of the most useful properties of J is stated in the following lemma.

Lemma 1.1. *Assume that the vector field u is C^1 smooth, let X be defined by* (1.1)–(1.2)*, and J be given by* (1.15)*. Then we have*

$$\partial_t J(a, t) = J(a, t) \left((\nabla \cdot u)(X(a, t), t) \right) \tag{1.16}$$

pointwise in (a, t).

Exercise 1.2. Obtain the evolution of the matrix $\nabla_a X$ and prove Lemma 1.1.

An immediate consequence of Lemma 1.1 is that $J \equiv 1$ if u is incompressible.

Corollary 1.3. *Under the assumption of Lemma 1.1, further assume that u is divergence free, i.e. that (1.14) holds. Then we have that*

$$J(a,t) = 1$$

for all $a \in \Omega$ and $t > 0$.

Proof of Corollary 1.3. From (1.2) it follows that $J(\cdot, 0) = \det(I) = 1$ identically. Solving the differential equation (1.16) we arrive at

$$J(a,t) = J(a,0) \exp\left(\int_0^t ((\nabla \cdot u)(X(a,s),s))\, ds \right) = J(a,0) = 1$$

by using that $\nabla \cdot u = 0$ identically. □

Another consequence of Lemma 1.1 is that we may compute the rate of change of the average of a quantity f over a domain $V(t)$ transported by the fluid.

Lemma 1.4. *Let $f \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a C^1 smooth function, and assume that the velocity field u defining the flow map $X(\cdot, t)$ is also C^1. Then we have that*

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t) dx \right) = \int_{V(t)} \left(\partial_t f + \nabla \cdot (fu)\right)(x,t) dx \qquad (1.17)$$

for every $t > 0$ and every fluid element V.

Proof of Lemma 1.4. By using the change of variables formula, the convective derivative identity (1.4), and Lemma 1.1 we deduce

$$\begin{aligned}
\frac{d}{dt}\left(\int_{V(t)} f(x,t) dx \right) &= \frac{d}{dt}\left(\int_{V} f(X(a,t),t) J(a,t) da \right) \\
&= \int_V (D_t f)(X(a,t),t) J(a,t) da + \int_V f(X(a,t),t) \partial_t J(a,t) da \\
&= \int_V (\partial_t f + u \cdot \nabla f + (\nabla \cdot u) f)(X(a,t),t) J(a,t) da \\
&= \int_V (\partial_t f + \nabla \cdot (uf))(X(a,t),t) J(a,t) da \\
&= \int_{V(t)} (\partial_t f + \nabla \cdot (uf))(x,t) dx
\end{aligned}$$

which concludes the proof. □

Note that if we set $f = \rho$ in the above lemma, then the above yields another proof of (1.11). Another particular case is to let $f = 1$. In that case the conservation of volume assumption (1.13) yields that

$$0 = \frac{d}{dt}|V(t)| = \frac{d}{dt}\left(\int_{V(t)} 1 dx \right) = \int_{V(t)} (\nabla \cdot u)(x,t) dx$$

for every $t \geq 0$ and every open set $V \subset \Omega$. Since $X(\cdot, t)$ is a bijection, it follows that

$$\int_W (\nabla \cdot u)(x,t)dx = 0$$

for every open set $W \subset \Omega$ and all $t \geq 0$, from which (1.14) immediately follows.

Remark 1.5 (Mass density in an incompressible fluid). At this stage we note that incompressibility has a powerful effect on the evolution of the density ρ: this density is transported along particle trajectories. Indeed, combining (1.12) with (1.14) we arrive at

$$0 = \partial_t \rho + u \cdot \nabla \rho + \rho(\nabla \cdot u) = D_t \rho,$$

from which it follows from (1.4) that

$$\partial_t \rho(X(a,t),t) = 0$$

pointwise in (a,t). We thus obtain the identities

$$\rho(X(a,t),t) = \rho(a) \qquad \text{or equivalently} \qquad \rho(x,t) = \rho_0(A(x,t)), \tag{1.18}$$

where $\rho_0(x) = \rho(x,0)$ is the initial mass density configuration. In particular, if the initial density is *homogenous*, i.e.

$$\rho_0(x) = \rho_0, \text{ a constant,}$$

for all $x \in \Omega$, then we obtain from (1.18) that

$$\rho(x,t) = \rho_0 \tag{1.19}$$

for all $x \in \Omega$ and all $t \geq 0$.

1.6 Conservation of momentum in an ideal fluid

The assumption of an *ideal* fluid is that if S is any surface in the fluid with a chosen outward unit normal n, then the force of stress exerted on this surface, per unit area, at the point $x \in S$ is given by a pressure density

$$p(x,t)n(x).$$

The units of p are $[p] = ML^{-2} \cdot LT^{-2}$, i.e. it expresses the acceleration of mass per unit area. Equivalently, if $S = \partial V$ is the boundary of a fluid element V in the fluid, the *force of stress* acting on the fluid volume V is only coming in through its boundary, and is given by

$$-\int_{\partial V} p(x,t)n(x)dx = -\int_V \nabla p(x,t)dx$$

via the Gauss/Green theorem.

The *conservation of momentum, is an expression of Newton's second law of motion*:

$$\frac{d}{dt}\left(\int_{V(t)} \rho(x,t)u(x,t)dx\right) = -\int_{V(t)} \nabla p(x,t)dx \tag{1.20}$$

where V is an arbitrary volume in the ideal fluid. In order to derive the differential equation associated to the conservation of momentum we need to further compute the left side of (1.20) above.

The proof of Lemma 1.4 combined with the conservation of mass assumption (1.12) gives the following useful result.

Theorem 1.6 (Transport theorem)**.** *Assume f, ρ are C^1 smooth, and that ρ obeys* (1.12)*. Then we have that*

$$\frac{d}{dt}\left(\int_{V(t)} \rho(x,t) f(x,t) dx\right) = \int_{V(t)} \rho(x,t)(D_t f)(x,t) dx \tag{1.21}$$

holds for all $t \geq 0$ and all open sets $V \subset \Omega$.

Proof of Theorem 1.6. Replacing f by ρf in Lemma 1.4, and using (1.12), we arrive at

$$\begin{aligned}\frac{d}{dt}\left(\int_{V(t)} \rho(x,t) f(x,t) dx\right) &= \int_{V(t)} (\partial_t(\rho f) + \nabla \cdot (\rho u f))(x,t) dx \\ &= \int_{V(t)} (f \partial_t \rho + f \nabla \cdot (\rho u))(x,t) dx + \int_{V(t)} (\rho \partial_t f + \rho u \cdot \nabla f)(x,t) dx \\ &= 0 + \int_{V(t)} (\rho D_t f)(x,t) dx\end{aligned}$$

which concludes the proof. □

A corollary of the Transport theorem is that we may compute the rate of change of the density in a moving fluid element. Setting $f = u$ in (1.21) we arrive at

$$\frac{d}{dt}\left(\int_{V(t)} \rho(x,t) u(x,t) dx\right) = \int_{V(t)} \rho(x,t)(D_t u)(x,t) dx$$

which combined with (1.20) yields the integral form of the conservation of momentum in an ideal fluid:

$$\int_{V(t)} \rho(x,t)(\partial_t u + u \cdot \nabla u)(x,t) dx = -\int_{V(t)} \nabla p(x,t) dx \tag{1.22}$$

for any volume V and time $t \geq 0$. Assuming all quantities are smooth we thus arrive at the vector differential equation

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p \tag{1.23}$$

pointwise at (x,t) in the fluid.

Remark 1.7 (Bulk force). If in addition to the force coming from the internal pressure in the fluid we wish to incorporate the effect of a body/bulk force $f(x,t)$ acting at the point x in the fluid, per unit mass, then we have

$$\frac{d}{dt}\left(\int_{V(t)} \rho(x,t) u(x,t) dx\right) = -\int_{\partial V(t)} p(x,t) n(x) dx + \int_{V(t)} \rho(x,t) f(x,t) dx$$

and thus (1.23) becomes

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + \rho f$$

with the units of measurement of the force per unit mass being $[f] = LT^{-2}$.

1.7 The Euler equation for an ideal homogenous incompressible fluid

Summarizing the equation for the conservation of momentum (cf. (1.19)) for a homogenous incompressible fluid (cf. (1.14)) , and the conservation of mass when the fluid is ideal (cf. (1.23)), we arrive at the *homogenous incompressible Euler equations*

$$\partial_t u + u \cdot \nabla u = -\frac{1}{\rho_0}\nabla p + f \tag{1.24}$$

$$\nabla \cdot u = 0 \tag{1.25}$$

where $\rho_0 \neq 0$ is a constant (in space and time) density, and f is a bulk force. In order to study the Cauchy problem the system is supplemented with an initial condition

$$u_0(x) = u(x, 0). \tag{1.26}$$

For simplicity of notation, throughout these notes we normalize the constant density ρ_0 to equal 1, and correspondingly redefine the pressure as

$$\rho_0 \mapsto 1 \qquad \text{and} \qquad \frac{p(x,t)}{\rho_0} \mapsto p(x,t).$$

The only aspect we need to be careful about in this renormalization, is that the units of the normalized pressure have now become $[p] = L^2T^{-2}$. The homogenous incompressible Euler equations thus read

$$\partial_t u + u \cdot \nabla u = -\nabla p \tag{1.27}$$

$$\nabla \cdot u = 0 \tag{1.28}$$

where we have for the moment ignored any effects due to a body forcing term f. Note that the units of all the terms in the moment equation (1.27) are LT^{-2}, i.e. units of acceleration. Written in component form, and using the Einstein summation convention on repeated indices, the system (1.27)–(1.28) reads

$$\partial_t u_j + u_i \partial_i u_j = -\partial_j p \tag{1.29}$$

$$\partial_i u_i = 0 \tag{1.30}$$

for all $j \in \{1, \ldots, d\}$.

1.8 Boundary conditions

So far we have not paid attention to the domain $\Omega \subset \mathbb{R}^d$ on which the Euler system (1.27)–(1.28) is considered, and thus we have also not been careful about what happens on $\partial\Omega$. In these notes we shall consider the following types of domains Ω and boundary conditions.

1.8.1 The whole space $\Omega = \mathbb{R}^d$

Here we shall informally assume that

$$|u(x,t)| + |p(x,t)| \to 0 \qquad \text{as} \qquad |x| \to \infty, \qquad \text{sufficiently fast.} \tag{1.31}$$

Rigorously, condition (1.31) shall be expressed in terms of integrability properties of u, p, and their derivatives.

1.8.2 The periodic domain $\Omega = \mathbb{T}^d$

Here we assume that both u and p are $\mathbb{T}^d$-periodic functions, with zero mean on $\mathbb{T}^d = [-\pi, \pi]^d$, i.e.

$$\int_{\mathbb{T}^d} u(x,t)dx = \int_{\mathbb{T}^d} p(x,t)dx = 0. \tag{1.32}$$

Note that the pressure is defined up to a function of time, so that this normalization of p shall ensure the uniqueness of p. On the other hand, for the velocity field we have that the mean is a constant of motion, i.e. from (1.27)–(1.28) we deduce

$$\frac{d}{dt}\left(\int_{\mathbb{T}^d} u(x,t)dx\right) = 0,$$

so that if the initial datum u_0 has zero mean, so does the solution $u(\cdot,t)$ at all $t \geq 0$.

1.8.3 $\Omega \subset \mathbb{R}^d$ open, simply connected, with smooth boundary

Here we assume that no fluid is "leaving the domain" Ω, i.e. that

$$u(x,t) \cdot n(x) = 0 \qquad \text{for} \qquad x \in \partial\Omega. \tag{1.33}$$

This boundary condition is sometimes referred to as a *slip boundary condition*, or *non-penetrating boundary condition*. It expresses the fact that there is no tangential friction between the fluid and the boundary.

We note that other boundary conditions may also be considered so that the Euler equations are well-posed, but for the moment we shall focus on the ones listed above, or a combination of them (e.g. periodicity in x_1, and decay at spacial infinity with respect to x_2).

1.9 The pressure

As may be better seen in component the form (1.29)–(1.30), the Euler system consists of $d+1$ differential equations, for the $d+1$ unknowns $(u,p) = (u_1, \ldots, u_d, p)$. Thus the fact that there is no evolution equation for the pressure is not problematic: the pressure may be recovered from the velocity field. In fact the pressure p should be understood a Lagrange multiplier associated to the incompressibility constraint.

In order to compute p we apply the divergence operator to (1.27), and use the incompressibility constraint in the form $\nabla \cdot \partial_t u = \partial_t(\nabla \cdot u) = 0$, to obtain

$$\begin{aligned} -\Delta p &= \nabla \cdot ((u \cdot \nabla)u) = \partial_j(u_i \partial_i u_j) \\ &= \partial_{ij}(u_i u_j). \end{aligned} \tag{1.34}$$

In the last equality above we have again appealed to $\nabla \cdot u = 0$. Note that one may also write (1.34) as

$$-\Delta p = \partial_j u_i \, \partial_i u_j \tag{1.35}$$

with the summation convention on repeated indices being used throughout. Thus, in order to solve for the pressure, we need to invert the Laplacian, which in turn requires boundary conditions (e.g. note that from (1.34) alone, the pressure p is only defined up a harmonic function).

1.9.1 The whole space $\Omega = \mathbb{R}^d$

Inverting the Laplacian on the whole space, under decaying boundary conditions at spacial infinity amounts to convolving with the Newtonian potential (see e.g. [Eva98] for further details).

Theorem 1.8 (Inverting the Laplacian on $\mathbb{R}^d$)**.** *Assume that $f \in C_c^2(\mathbb{R}^d)$, and define*

$$\varphi(x) = \int_{\mathbb{R}^d} N(y) f(x-y)dy \tag{1.36}$$

where

$$N(y) = \begin{cases} -\frac{1}{2\pi}\log|y|, & d=2, \\ \frac{1}{d(d-2)\alpha_d}|y|^{2-d}, & d \geq 3, \end{cases} \tag{1.37}$$

is the Newtonian potential, where $\alpha_d = 2\pi^{d/2}/(d\Gamma(d/2))$ gives the volume of $\mathbb{S}^{d-1}$. Then we have that $\varphi \in C^2(\mathbb{R}^d)$, and that

$$-\Delta\varphi(x) = f(x) \tag{1.38}$$

for all $x \in \mathbb{R}^d$. Assuming further that $|\varphi| \to 0$ as $|x| \to \infty$, the representation formula (1.36) yields the unique solution to the Poisson equation (1.38).

The decay as $|x| \to \infty$ ensures that the harmonic function up to which the pressure is defined is in fact the 0 function. It is clear that the assumption on f in Theorem 1.8 is sufficient, but not necessary: it may be relaxed so that the convolution with the Newtonian potential defines an L^1_{loc} function, and so that the Poisson equation $-\Delta\varphi = f$ holds in the sense of distributions.

Inserting $f = \partial_{ij}(u_i u_j)$ for f in (1.36), and integrating by parts with respect to y_i and y_j, we arrive at:

Theorem 1.9 (Pressure on the whole space)**.** *Assume that $u \in L^{2q}(\mathbb{R}^d)$ for some $q \in (1,\infty)$. Then the equation (1.34) has a unique solution $p \in L^q(\mathbb{R}^d)$, which is given explicitly as*

$$p(x) = -\frac{\delta_{ij}}{d}u_i(x)u_j(x) + p.v.\int_{\mathbb{R}^d} K_{ij}(y)u_i(x-y)u_j(x-y)dy \tag{1.39}$$

where the kernel K_{ij} is of Calderón-Zygmund type and is given explicitly as

$$K_{ij}(y) = \frac{y_i y_j - \frac{\delta_{ij}}{d}|y|^2}{\alpha_d |y|^{d+2}}, \tag{1.40}$$

δ_{ij} denotes the usual Kronecker symbol, and the summation convention on repeated indices is used.

Remark 1.10 (Estimates for the pressure). Using the boundedness of Calderón-Zygmund operators on L^q spaces (see [Ste70]), under the assumptions of Theorem 1.9 we have that

$$\|p\|_{L^q(\mathbb{R}^d)} \lesssim_d \frac{q^2}{q-1}\|u\|^2_{L^{2q}(\mathbb{R}^d)} \tag{1.41}$$

for all $q \in (1,\infty)$, with the endpoint case $q = \infty$ yielding $p \in BMO$. Moreover, using the Gagliardo-Nireneberg-Sobolev inequalities (see [Eva98]) we may infer that if $u \in W^{s,q}(\mathbb{R}^d)$, with $s > 1 + d/q$, then the estimate

$$\|\nabla p\|_{W^{s,q}(\mathbb{R}^d)} \lesssim_{d,s,q} \|u\|^2_{W^{s,q}(\mathbb{R}^d)} \tag{1.42}$$

holds. Note that to obtain the above bound (1.42), one needs to explore that u is divergence free once more, i.e. to use the right side of the Poisson equation given by (1.35) instead of (1.34). The condition $s > 1 + d/q$ ensures that $\nabla u \in L^\infty$ for all $u \in W^{s,p}$.

Exercise 1.11. Assuming the necessary Sobolev-type inequalities, prove estimate (1.42).

A direct and equivalent way to to solve (1.34) is to use the Fourier transform

$$\widehat{\varphi}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) e^{i\xi\cdot x} dx$$

which when applied to (1.34)

$$|\xi|^2 \widehat{p}(\xi) = -\xi_i \xi_j \widehat{(u_i u_j)}(\xi) = -\xi_i \xi_j \int_{\mathbb{R}^d} \widehat{u}_i(\eta) \widehat{u}_j(\xi - \eta) d\eta.$$

Thus, after dividing by $|\xi|^2$ and applying the inverse Fourier transform, we arrive at

$$p(x) = \left(\frac{-\xi_i \xi_j}{|\xi|^2} \widehat{(u_i u_j)}(\xi) \right)^{\vee} (x) = R_i R_j (u_i u_j)(x) \tag{1.43}$$

where the operators R_k with $k \in \{1, \dots, d\}$ are the Riesz transforms (multi-dimensional analogues of the Hilbert transform). These operators are given either as a Fourier multiplier with symbol $i\xi_k/|\xi|$, i.e.

$$\widehat{R_k f}(\xi) = \frac{i\xi_k}{|\xi|} \widehat{f}(\xi)$$

or as a convolution with a Calderón-Zygmund kernel K_k

$$\begin{aligned} R_k f(x) &= p.v. \int_{\mathbb{R}^d} K_k(y) f(x-y) dy \\ &= \frac{1}{\pi \alpha_{d-1}} p.v. \int_{\mathbb{R}^d} \frac{y_k}{|y|^{d+1}} f(x-y) dy \end{aligned}$$

where the principal value is understood as $|y| \to 0$ and if f does not decay sufficiently fast also as $|y| \to \infty$. The formulas (1.39)–(1.40) now follow immediately if we consider the convolutions kernels associated with iterated Riesz transforms $R_i R_j$. These kernels consist of the Calderón-Zygmund kernel K_{ij} defined above, and for $i = j$ they additionally contain a Dirac mass, responsible for the first term on the right side of (1.39) (this is why we have the identity $R_1 R_1 + \ldots + R_d R_d = -1$). See [Ste70] for details.

1.9.2 The periodic domain $\Omega = \mathbb{T}^d$

Inverting the Laplacian on the periodic domain may again be simply achieved by dividing by the square of the length of the frequency vector $k \in \mathbb{Z}^d$. This division is justified if the $k = 0$ frequency is absent, i.e. if the function has zero mean (recall (1.32)). Thus, similarly to (1.43) we have that $\widehat{p}(0) = 0$, and

$$\widehat{p}(k) = \frac{-k_i k_j}{|k|^2} \widehat{(u_i u_j)}(k) = \frac{-k_i k_j}{|k|^2} \sum_{\ell \in \mathbb{Z}^d \setminus \{0,k\}} \widehat{u}_i(\ell) \widehat{u}_j(k - \ell)$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$ is the unique solution of (1.34) with boundary condition (1.32). Recall that the summation convention over the repeated indices $i, j \in \{1, \dots, d\}$ is used above and throughout these notes. The equivalent formulas to (1.39)–(1.40) may be obtained by periodizing the kernels and using Poisson summation. More importantly, the bounds (1.41)–(1.42) also hold in the periodic case.

1.9.3 $\Omega \subset \mathbb{R}^d$ open, simply connected, with smooth boundary

Solving the Poisson equation (1.34) in a bounded domain requires boundary conditions, typically Dirichlet or Neumann. In the case of the Euler equations, the boundary conditions for p may be obtained by taking the dot product to (1.27) with the outward unit normal n to $\partial\Omega$, to obtain

$$\partial_t u_j n_j + u_i \partial_i u_j n_j = -\partial_j p n_j, \qquad \text{on } \partial\Omega.$$

At this stage, we appeal to the slip boundary condition

$$u \cdot n = u_j n_j = 0, \qquad \text{on } \partial\Omega,$$

to obtain (since the domain is not moving with time)

$$\frac{\partial p}{\partial n} = \nabla p \cdot n = -((u \cdot \nabla)u) \cdot n, \qquad \text{on } \partial\Omega. \tag{1.44}$$

Thus, the pressure obeys Neumann boundary conditions at $\partial\Omega$.

In order to solve the Neumann problem

$$\begin{cases} \Delta\varphi = f, & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} = g, & \text{on } \partial\Omega, \end{cases} \tag{1.45}$$

the functions f and g must obey the compatibility condition

$$\int_\Omega f dx = \int_{\partial\Omega} g dx$$

in view of the divergence theorem applied to the vector field $\nabla\varphi$. In our case cf. (1.34) we have $f = -\nabla \cdot (u \cdot \nabla u)$ and $g = -(u \cdot \nabla u) \cdot n$, cf. (1.44), so that the compatibility condition automatically holds in view of the divergence theorem. Thus, we may solve for p the Neumann problem for the Poisson equation (1.34), (1.44). Note that uniqueness does not hold as p is only defined up to a constant function of space. This is however not a problem, because ∇p (not p itself) appears in (1.27).

Exercise 1.12. Solve explicitly the Neumann problem for the pressure (1.34), (1.44) in the case of the half space $\Omega = \mathbb{R}^d_+$. That is, obtain a formula similar to (1.39) in the case of the half space. For this purpose use the explicit Greens function for the half space in terms of the Newtonian potential and integrate twice by parts, taking into account the boundary conditions.

Remark 1.13 (Bounds for the pressure gradient on a bounded domain). In order to obtain good estimates for the pressure gradient in a bounded domain, i.e. the same bound as (1.42), we need to use once more that that $\nabla \cdot u = 0$ in Ω and that $u \cdot n = 0$ on $\partial\Omega$. This allows us to rewrite the problem (1.34), (1.44) as

$$-\Delta p = \partial_i u_j \, \partial_j u_i, \qquad \text{in } \Omega, \tag{1.46}$$

$$\frac{\partial p}{\partial n} = u_i u_j \phi_{ij}, \qquad \text{on } \partial\Omega, \tag{1.47}$$

where ϕ_{ij} are smooth functions that depend solely on the local slope and curvature of Ω. More precisely, consider a neighborhood of a point $x \in \partial\Omega$, in which $\partial\Omega$ is the level set of a smooth function ψ, i.e. locally near x we have $\partial\Omega = \{y \in \mathbb{R}^d \colon \psi(y) = 0\}$. Then we have

$$\phi_{ij}(x) = \frac{\partial_{ij}\psi(x)}{|\nabla\psi(x)|}. \tag{1.48}$$

The equation in the interior of the domain is nothing but (1.35). The equation for the boundary follows from [Tem75]. The desired estimate

$$\|\nabla p\|_{W^{s,q}(\Omega)} \lesssim_{d,s,q,\Omega} \|u\|^2_{W^{s,q}(\Omega)} \tag{1.49}$$

now follows from (1.46)–(1.47) and the classical work [ADN59] in the case of a general smooth domain Ω.

Exercise 1.14. Prove that (1.44) may indeed be rewritten as (1.48).

1.10 The Leray projector

The *Leray projector*, henceforth denoted by $\mathbb{P}$, is an orthogonal projection from L^2 onto the closed subset of divergence free vector fields in L^2:

$$L^2_\sigma(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \colon \nabla \cdot u = 0\} \tag{1.50}$$

$$L^2_\sigma(\mathbb{T}^d) = \left\{u \in L^2(\mathbb{R}^d) \colon \nabla \cdot u = 0, \int_{\mathbb{T}^d} u dx = 0\right\} \tag{1.51}$$

$$L^2_\sigma(\Omega) = \{u \in L^2(\mathbb{R}^d) \colon \nabla \cdot u = 0 \text{ in } \Omega, u \cdot n = 0 \text{ on } \partial\Omega\}.$$

The Leray projector is intimately related to the Helmholtz-Hodge decomposition of vector fields with components in $L^2(\mathbb{R}^d)$:

Theorem 1.15 (Helmholtz-Hodge decomposition)**.** *Let Ω be either $\mathbb{R}^d$, $\mathbb{T}^d$, or a bounded simply-connected Lipschitz domain, with $d \in \{2,3\}$. Then we may write*

$$L^2(\Omega) = \{\nabla\varphi \colon \phi \in H^1(\Omega)\} \oplus \{\nabla \times \psi \colon \psi, \nabla \times \psi \in L^2(\Omega)\}$$

where in two dimensions the curl *operator $\nabla\times$ needs to be replaced by the rotated gradient operator $\nabla^\perp\cdot$, with $\nabla^\perp = (-\partial_2, \partial_1)$.*

1.10.1 The whole space $\Omega = \mathbb{R}^d$

For the whole space, the Leray projector may be defined as a Fourier multiplier operator. Let $\varphi \in L^2(\mathbb{R}^d)$ be a vector field. We wish to write

$$\varphi = \mathbb{P}\varphi + \nabla p$$

for some $p \in \dot{H}^1(\mathbb{R}^d)$. As earlier, since we expect $\mathbb{P}\varphi \in L^2_\sigma$, upon taking the divergence of the above we obtain $-\Delta p = -\nabla \cdot \varphi$. We thus obtain $\nabla p = -\nabla(-\Delta)^{-1}\nabla \cdot \varphi$, and thus the Leray projector is given by the matrix

$$\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{-1}\nabla \cdot .$$

Recalling (1.43), and switching to the Fourier variables, one then defines the kth component of the projection operator as

$$\widehat{(\mathbb{P}\varphi)}_k(\xi) = \widehat{\varphi}_k(\xi) - \frac{\xi_k\xi_j}{|\xi|^2}\widehat{\varphi}_j(\xi) \tag{1.52}$$

or equivalently write the Fourier symbol of $\mathbb{P}$ as

$$\widehat{\mathbb{P}}(\xi) = 1 - \frac{\xi \otimes \xi}{|\xi|^2} \tag{1.53}$$

for all $\xi \neq 0$. In order to verify that $\nabla \cdot \mathbb{P}\varphi = 0$, we simply note hat

$$\begin{aligned}(\widehat{\nabla \cdot \mathbb{P}\varphi})(\xi) = i\xi_k \widehat{(\mathbb{P}\varphi)}_k(\xi) &= i\xi_k \widehat{\varphi}_k(\xi) - \frac{i\xi_k\xi_k\xi_j}{|\xi|^2}\widehat{\varphi}_j(\xi) \\ &= (\widehat{\nabla \cdot \varphi})(\xi) - (\widehat{\nabla \cdot \varphi})(\xi) = 0.\end{aligned}$$

The fact that $\mathbb{P}$ is a bounded operator on L^2 follows from the boundedness of the symbol in (1.53). Moreover, it is clear that in the absence of boundaries $\mathbb{P}$ commutes with differentiation operators, and extends to a bounded operator from $\dot{H}^s(\mathbb{R}^d)$ to $\dot{H}^s_\sigma(\mathbb{R}^d) = L^2_\sigma(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ for any $s \geq 0$.

Note that if φ is a gradient, that is $\varphi = \nabla p$ for some $p \in H^1(\mathbb{R}^d)$, then

$$\widehat{(\mathbb{P}\varphi)}_k(\xi) = (\widehat{\mathbb{P}(\nabla p)})_k(\xi) = i\xi_k\widehat{p}(\xi) - \frac{\xi_k\xi_j}{|\xi|^2} i\xi_j\widehat{p}(\xi) \tag{1.54}$$

$$= i\xi_k\widehat{p}(\xi) - i\xi_k\widehat{p}(\xi) = 0, \tag{1.55}$$

that is, $\mathbb{P}$ annihilates gradients. Moreover, note that if φ is already divergence free, then $\mathbb{P}\varphi = \varphi$, as the second term in (1.52) vanishes.

Remark 1.16 (Leray-projected Euler equation). In view of these observations, and the fact that one may write the Euler equations as

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) = 0 \tag{1.56}$$

with a smooth initial datum $u_0 \in L^2_\sigma(\mathbb{R}^d)$. Indeed, applying $\mathbb{P}$ to (1.27) we obtain (1.56). Conversely, the evolution (1.56) ensures that the solution lies in $L^2_\sigma(\mathbb{R}^d)$ for positive time, and thus (1.28) is satisfied. Defining the pressure gradient ∇p as $(\mathrm{Id} - \mathbb{P})(u \cdot \nabla u)$, we also ensure that (1.27) is obeyed.

1.10.2 The periodic domain $\Omega = \mathbb{T}^d$

On the periodic domain, the Leray projector is defined as in the whole space, in terms of its Fourier multiplier operator (1.52)–(1.53), upon replacing $\xi \in \mathbb{R}^d$ with $k \in \mathbb{Z}^d$. In order to ensure that $\int_{\mathbb{T}^d} \mathbb{P}\varphi dx = 0$, we set $\widehat{\mathbb{P}}(0) = 0$, which ensures that $\mathbb{P}\colon L^2(\mathbb{T}^d) \to L^2_\sigma(\mathbb{T}^d)$.

1.10.3 $\Omega \subset \mathbb{R}^d$ open, simply connected, with smooth boundary

In view of the discussion regarding the Helmholtz-Hodge decomposition for the whole space, in the case of a simply connected smooth bounded domain Ω, it is natural to define the operator $\mathrm{Id} - \mathbb{P}$, rather than $\mathbb{P}$ itself, by

$$(\mathrm{Id} - \mathbb{P})\varphi = \nabla p$$

where p is the zero mean solution of the Neumann problem (1.45), in which we set

$$f = \nabla \cdot \varphi, \qquad \text{and} \qquad g = \varphi \cdot n.$$

By construction, we clearly have that the image of $\mathbb{P}$ lies in the subset of $L^2_\sigma(\Omega)$. Also in this case, $\mathbb{P}$ may be extended as a bounded linear operator on $W^{s,p}(\Omega)$, which follows from the regularity results for the Neumann problem (if $\varphi \in W^{s,p}(\Omega)$, then $f \in W^{s-1,p}(\Omega)$, and by the trace theorem $g \in W^{s-1/p,p}(\partial\Omega)$) available in [ADN59].

1.11 Vorticity and the Biot-Savart law

The *vorticity* in a fluid whose velocity field is u is the vector field

$$\begin{aligned} \omega &= (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1) \\ &= \nabla \times u. \end{aligned} \tag{1.57}$$

The geometric meaning of the vorticity is that it gives the axis and strength of local rotation in the fluid flow. This may be seen from the Taylor formula

$$\begin{aligned} u(x+h) &= u(x) + \nabla u(x) \cdot h + O(|h|^2) \\ &= u(x) + D_u(x) \cdot h + \Omega_u(x) \cdot h + O(|h|^2) \end{aligned}$$

for $0 < |h| \ll 1$, and we have denoted by D the symmetric part of the velocity gradient matrix

$$D_u(x) = \frac{(\nabla u)(x) + (\nabla u)^T(x)}{2}$$

and by Ω the anti-symmetric part of this matrix

$$\Omega_u(x) = \frac{(\nabla u)(x) - (\nabla u)^T(x)}{2}.$$

Then we have that

$$\Omega_u(x) \cdot h = \frac{1}{2}\omega(x) \times h$$

so that locally near x the action of Ω_u is given by rigid body rotation around the axis

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}.$$

On the other hand, D_u is symmetric, thus diagonalizable, and in a corresponding orthonormal basis the action of D_u on a vector h is to stretch the the components of h in this basis exponentially by the eigenvalues of D_u. Note that the sum of these eigenvalues is the trace of $D_u(x)$, which is nothing but $\nabla \cdot u = 0$.

1.11.1 Vorticity formulation of the Euler equations in three dimensions

Using the vector identities

$$\begin{aligned} (u \cdot \nabla) u &= \omega \times u + \frac{1}{2}\nabla(|u|^2) \\ \nabla \times \nabla \left(p + \frac{1}{2}|u|^2\right) &= 0 \\ \nabla \times (\omega \times u) &= \omega(\nabla \cdot u) - u(\nabla \cdot \omega) + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u \end{aligned}$$

upon taking the curl of (1.27) and using incompressibility (1.28) we obtain the evolution of the Euler vorticity as

$$\partial_t \omega + (u \cdot \nabla \omega) = (\omega \cdot \nabla) u. \tag{1.58}$$

Note that in view of the decomposition of ∇u into its symmetric D_u and antisymmetric Ω_u part, using that $\omega \times \omega = 0$, we obtain

$$(\omega \cdot \nabla)u = D_u \cdot \omega,$$

i.e. only the symmetric part of the velocity gradient is stretching the vorticity.

It turns out that the system (1.58) is in fact equivalent to the Euler equations in velocity form (1.27)–(1.28) (at least for smooth solutions), as one may compute the velocity field u from the vorticity ω, using the *Biot-Savart law*. This amounts to solving the div-curl system

$$\begin{cases} \nabla \cdot u = 0 \\ \nabla \times u = \omega \end{cases} \tag{1.59}$$

supplemented with one of the three boundary conditions (1.31), (1.32), (1.33) (for u, there are no boundary conditions for ω).

We will focus first on the case of the whole space, $\Omega = \mathbb{R}^d$. Applying the curl operator to the second equation in (1.59) we arrive at

$$\nabla \times \omega = \nabla \times (\nabla \times u) = -\Delta u + \nabla(\nabla \cdot u) = -\Delta u.$$

Thus, we may invert the Laplacian and obtain that the velocity field is

$$u = (-\Delta)^{-1}\nabla \times \omega = \nabla \times (-\Delta)^{-1}\omega = \nabla \times \psi \tag{1.60}$$

where ψ is called the *stream function*. Note that with this definition we have $\nabla \cdot u = \nabla \cdot (\nabla \times \psi) = 0$, and $\nabla \times u = -\Delta\psi + \nabla(\nabla \cdot \psi) = \omega$. Thus (1.60) defines a solution to (1.59). The uniqueness of these solutions are guaranteed by the decaying boundary condition, which ensures that the only harmonic function obeying (1.31) is the trivial one.

In Fourier variables, we have the explicit formula

$$\widehat{u}(\xi) = \frac{i\xi}{|\xi|^2} \times \widehat{\omega}(\xi),$$

while appealing to Theorem 1.8 and integrating by parts the curl operator onto the Newtonian potential we arrive at the *Biot-Savart law for* $\mathbb{R}^3$:

$$u(x) = \int_{\mathbb{R}^3} K_3(x-y) \times \omega(y)dy \tag{1.61}$$

where the 3D Biot-Savart kernel is defined as

$$K_3(y) = \frac{y}{4\pi|y|^3} \tag{1.62}$$

for all $y \neq 0$.

Remark 1.17 (The periodic case). In the case of periodic boundary conditions, we still have that (1.60) defines the unique solution of the div-curl system (1.59), the only difference is given by inverting the Laplacian. In Fourier variables, this is achieved by merely replacing $\xi \in \mathbb{R}^3$ with $k \in \mathbb{Z}^3$. In real-variables, the convolution is agains the periodic Biot-Savart kernel, given by (up to a multiplicative constant?)

$$K_{3,\text{per}}(y) = \sum_{k\in\mathbb{Z}^3} K_3(y - 2\pi k)$$

for all $y \in \mathbb{T}^d \setminus \{0\}$.

Remark 1.18 (The bounded domain). In order to obtain the Biot-Savart law for the bounded domain, we need to again invert the Poisson equation for the stream function $-\Delta\psi = \omega$, and then merely define $u = \nabla \times \psi$. The question which arises is what boundary conditions to enforce on ψ. Using the slip boundary condition for the velocity, it turns out that the answer is Dirichlet: the condition $0 = u \cdot n = (\nabla \times \psi) \cdot n$ is ensured if we let $\psi = \text{consntat}$ on $\partial\Omega$. For simplicity, this constant is set to equal 0. This construction also ensures that one has a unique solution ψ, and thus u, in terms of ω.

Exercise 1.19. Obtain the explicit Biot-Savart law of the half-space $\mathbb{R}^3_+$.

At this stage we note that he velocity is indeed one derivative smoother than the vorticity. Indeed, in Fourier variables we have

$$\widehat{(\nabla u)}(\xi) = -\left(\frac{\xi}{|\xi|} \times \widehat{\omega}(\xi)\right) \otimes \frac{\xi}{|\xi|} \tag{1.63}$$

which is a zero order homogenous multiplier acting on ω, mapping $L^p \to L^p$ boundedly when $p \in (1, \infty)$, and $H^s \to H^s$ boundedly, for all $s \geq 0$. Alternatively, we may carefully differentiate (1.61) with respect to x, we obtain that the action of the ∇u matrix on a vector h is given by

$$\begin{aligned}\nabla u(x)h &= \frac{1}{3}\omega(x) \times h \\ &\quad - \frac{1}{4\pi} p.v. \int_{\mathbb{R}^3} \frac{1}{|x-y|^3} \left(\omega(y) \times h + 3\left(\frac{x-y}{|x-y|} \times \omega(y)\right) \frac{x-y}{|x-y|} \cdot h\right) dy.\end{aligned} \tag{1.64}$$

We notice that the integral defining the second term in (1.64) is a vector Calderón-Zygmund operator of convolution type, which holds since

$$3\int_{\mathbb{S}^2} (z \times \omega) z \cdot h dz = -\frac{4\pi}{3}\omega \times h = -\int_{\mathbb{S}^2} \omega \times h dz$$

for any fixed vectors ω and h.

Exercise 1.20. Recall that the stretching term on the right side of (1.58) may be written solely in terms of the symmetric part of the velocity gradient matrix as $\omega \cdot \nabla u = D_u \cdot \omega$. Similarly to (1.64), show that the action of D_u on a vector h, i.e. $D_u \cdot h$, is given by the convolution with a Calderón-Zygmund kernel, and find this kernel explicitly. Furthermore, find a singular integral expression for the quantity

$$\alpha(x) := (D_u(x) \cdot \xi(x)) \cdot \xi(x)$$

solely in terms of ξ and $|\omega|$, where

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}$$

is the vorticity direction. The motivation comes from the fact that the magnitude of the vorticity $|\omega|$ in 3D Euler (1.58) obeys

$$(\partial_t + u \cdot \nabla)|\omega| = \alpha|\omega|.$$

This expression may be used to show that vorticity alignment depletes the nonlinearity [CF93, CFM96].

1.11.2 Vorticity formulation of the Euler equations in two dimensions

In some physical situations, e.g. for the flow around a very long cylinder whose axis is parallel to the x_3 axis, or to study the large scale dynamics of the Earth's atmosphere, to first order approximation we may consider the *two dimensional Euler equations*.

Here we assume that $u_3 \equiv 0$, and that u_1, u_2, and p are functions that depend solely on x_1 and x_2. Under these assumptions, cf. (1.57) the vorticity vector becomes

$$\omega = (0, 0, \bar{\omega})$$

where we have denoted by $\bar{\omega}$ the scalar vorticity

$$\bar{\omega} = \partial_1 u_2 - \partial_2 u_1 = \nabla^{\perp} \cdot u \tag{1.65}$$

and the vector which has this scalar vorticity as its third and only nontrivial component. Note that in this case the nonlinear term on the right side of (1.58) becomes

$$\omega \cdot \nabla u = \bar{\omega} \partial_3 u = 0 \tag{1.66}$$

since u does not depend on x_3. Thus, (1.58) becomes purely a transport equation for $\bar{\omega}$, and henceforth we shall abuse notation and denote $\bar{\omega}$ also simply as ω.

The two dimensional Euler equations in vorticity form thus become

$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{1.67}$$

In order to have a self-contained equation, we now need to solve for the velocity field u in terms of the scalar vorticity ω. As in 3D, this amounts to solving the div-curl system (1.59), in which the curl operator $\nabla \times$ is to be replaced by its 2D analogue: the rotated gradient vector $\nabla^{\perp} \cdot$. Similarly, we introduce a (scalar this time) stream function ψ which solves the Poisson equation

$$-\Delta \psi = \omega$$

with the decay boundary conditions on $\mathbb{R}^2$, the periodicity and zero mean conditions on $\mathbb{T}^2$, and the Dirichlet boundary condition for a simply connected domain $\Omega \subset \mathbb{R}^2$. The velocity field u is then obtained as

$$u = \nabla^{\perp} \psi,$$

and it may be easily verified that it is a solution to (1.59).

In Fourier variables, the *two dimensional Biot-Savart law* simply becomes

$$\widehat{u}(\xi) = \frac{i\xi^{\perp}}{|\xi|^2} \widehat{\omega}(\xi)$$

for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. The equivalent formula holds in the periodic case upon replacing ξ by $k \in \mathbb{Z}^2 \setminus \{0\}$. In real variables, analogously to (1.61)–(1.62), by using the explicit Newtonian potential for two dimensions (cf. (1.37)) and applying the perpendicular gradient operator to this kernel, we arrive at

$$u(x) = \int_{\mathbb{R}^2} K_2(x - y) \omega(y) dy \tag{1.68}$$

where

$$K_2(y) = \frac{y^{\perp}}{2\pi |y|^2} \tag{1.69}$$

for all $y \in \mathbb{R}^2 \setminus \{0\}$, is the two dimensional Biot-Savart kernel. The periodic Biot-Savart kernel is defined as before as

$$K_{2,\text{per}}(y) = \sum_{k \in \mathbb{Z}^2} K_2(y - 2\pi k)$$

for all $y \in \mathbb{T}^2 \setminus \{0\}$.

Exercise 1.21. Obtain the two-dimensional Biot-Savart law for the half-pace $\mathbb{R}^2_+$.

To conclude, as in the three dimensional case we note that ∇u is given by a Calderón-Zygmund operator acting on ω. This may be seen easily in Fourier variables since

$$\widehat{(\nabla u)}(\xi) = -\frac{\xi \otimes \xi^\perp}{|\xi|^2} \widehat{\omega}(\xi).$$

In real variables, similarly to (1.64) we have that the velocity gradient matrix is given by

$$\nabla u(x) = \frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2\pi} p.v. \int_{\mathbb{R}^2} \frac{1}{|x-y|^2} \sigma\left(\frac{x-y}{|x-y|}\right) \omega(y) dy \tag{1.70}$$

where

$$\sigma(z) = \begin{pmatrix} 2z_1 z_2 & z_2^2 - z_1^2 \\ z_1^2 - z_2^2 & -2z_1 z_2 \end{pmatrix}$$

has the property that

$$\int_{\mathbb{S}^1} \sigma(z) dz = 1.$$

Thus, the second term on the right side of (1.70) is again a classical Calderón-Zygmund operator of convolution type.

Exercise 1.22. Assume that $\omega \in L^2(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0,1)$. Use (1.64) or (1.70) to show that then we have $\nabla u \in L^2(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$. Similarly, show that $\mathbb{P}\colon L^2(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d)$ is a bounded operator.

1.12 Lagrangian vortex dynamics

Recall that we denote by $X(a,t)$ the solution (1.1)–(1.2), i.e. the Lagrangian particle trajectories associated to the vector field u.

1.12.1 The two dimensional case

For $d = 2$, the equation the Euler equations in vorticity form (1.67) show that

$$\omega(X(a,t),t) = \omega_0(a) \tag{1.71}$$

for all $a \in \Omega$ and $t > 0$. This is merely an expression of the fact that the vorticity is transported around the fluid along the Lagrangian paths X. This fact in turn yields more conservation laws in the Euler equations when in the two dimensional case.

An interesting upshot of (1.71) is that one may write a self-contained nonlocal equation for the Lagrangian paths. Indeed, by combining (1.1), Corollary 1.3, the two dimensional Biot-Savart law (1.68), and (1.71) we arrive at

$$\partial_t X(a,t) = \int_{\mathbb{R}^2} K_2(X(a,t) - X(b,t))\omega_0(b)db. \tag{1.72}$$

In the smooth regime, one may show that the Lagrangian ODE (1.72) is equivalent to the 2D Euler equations.

Exercise 1.23. Similarly to (1.72), show that $\partial_t(\nabla_a X)$ may be computed as a nonlocal singular integral in terms of $\omega_0, X(\cdot,t)$, and $\nabla_a X(\cdot,t)$. Note that higher order spacial derivative of X do not appear in this formula.

Exercise 1.24. Assume that $\omega_0 = \mathbf{1}_{\Omega_0}$ where $\Omega_0 \subset \mathbb{R}^2$ is a smooth, bounded, simply connected subset of $\mathbb{R}^2$. From (1.71) it follows that $\omega(t) = \mathbf{1}_{\Omega(t)}$, where $\Omega(t) = X(\Omega_0,t)$. Parametrize $\partial\Omega(t)$ as $\{(z_1(\alpha,t), z_2(\alpha,t) \colon \alpha \in \mathbb{T}\}$. From the above discussion it follows that the solution of 2D Euler equation at time t is fully determined by the map $z(t)\colon \mathbb{T} \to \mathbb{R}^2$. Use (1.72) to formally derive the equation obeyed by $\partial_t z$. This is sometimes referred to as the *contour dynamics equation*. Use the previous exercise to obtain the equation obeyed by the $\partial_t(\partial_\alpha z)$. Take special care in this second step, since ∇u is well-defined only when acting on vector fields tangent to $\partial\Omega$ (such as $\partial_\alpha z$).

1.12.2 The three dimensional case

In contrast, for $d = 3$ the vorticity is not conserved on Lagrangian paths. Instead, we have the vorticity transport formula

$$\omega(X(a,t),t) = (\nabla_a X)(a,t)\omega_0(a) \tag{1.73}$$

for all $a \in \Omega$ and $t > 0$. In order to prove (1.73), we note that at time $t = 0$ the identity holds (since $X(a,0) = a$), and thus it is sufficient to show that both sides of (1.73) obey the same differential equation. On the one hand, we have from (1.58) that

$$\partial_t\left(\omega(X(a,t),t)\right) = (D_t\omega)(X(a,t),t) = (\nabla_x u)(X(a,t),t)\omega(X(a,t),t).$$

On the other hand, we have from (1.1) and the chain rule that that

$$\partial_t\left((\nabla_a X)(a,t)\right) = (\nabla_x u)(X(a,t),t)(\nabla_a X)(a,t).$$

By subtracting the above and integrating the resulting ODE, we obtain that

$$\begin{aligned}\omega(X(a,t),t) - (\nabla_a X)(a,t)\omega_0(a) &= (\omega_0(X(a,0)) - (\nabla_a X)(a,0)\omega_0(a)) \exp\left(\int_0^t (\nabla_x u)(X(a,s),s)ds\right)\\ &= 0\end{aligned}$$

in view of the above consideration regarding the initial datum, thereby proving (1.73).

As in the two dimensional case, also in three dimensions we may write a self-contained nonlocal evolution equation for $X(a,t)$. By combining (1.1), Corollary 1.3, the three dimensional Biot-Savart law (1.61), and (1.73) we arrive at

$$\partial_t X(a,t) = \int_{\mathbb{R}^3} K_3(X(a,t) - X(b,t)) \times \left((\nabla_b X)(b,t)\omega_0(b)\right) db. \tag{1.74}$$

In the smooth regime, one may show that the Lagrangian ODE (1.74) is equivalent to the 3D Euler equations.

Exercise 1.25. Similarly to (1.74), show that $\partial_t(\nabla_a X)$ may be computed as a nonlocal singular integral in terms of $\omega_0, X(\cdot,t)$, and $\nabla_a X(\cdot,t)$, and that higher order spacial derivative of X do not arise in this formula.

1.13 The Kelvin circulation theorem and consequences

Consider a simple closed curve $C \subset \mathbb{R}^3$, and denote its image under the flow map $X(\cdot, t)$ by $C(t)$.

Theorem 1.26 (Kelvin circulation theorem)**.** *The circulation around the curve $C(t) = X(C, t)$ is conserved by the 3D Euler flow. That is,*

$$\frac{d}{dt}\Gamma_{C(t)} = \frac{d}{dt}\left(\int_{C(t)} u \cdot ds\right) = 0$$

for all $t > 0$, where u is a smooth solution of (1.27)–(1.28).

In particular, if the length of the curve $C(t)$ shrinks, the angular velocity must increase.

Proof of Theorem 1.26. Consider a parametrization $a = a(\lambda)$ of $C = C(0)$, where $\lambda \in \mathbb{T}$. The fact that C is closed amounts to the parametrization $a(\lambda)$ to be $\mathbb{T}$-periodic. As long as the velocity u is sufficiently smooth ($C_t C_x^1$ will do) up to time t, it follows that we have a parametrization

$$C(t) = \{X(a(\lambda), t) \colon \lambda \in \mathbb{T}\}$$

of the curve at time t. Fro the change of variable theorem, it thus follows that

$$\Gamma_{C(t)} = \int_{C(t)} u \cdot ds = \int_{\mathbb{T}} u(X(a(\lambda), t), t) \cdot \partial_\lambda \left(X(a(\lambda), t)\right) d\lambda.$$

Differentiating the above with respect to time and using (1.4) we arrive at

$$\frac{d}{dt}\Gamma_{C(t)} = \int_{\mathbb{T}} (D_t u)(X(a(\lambda), t), t) \cdot \partial_\lambda \left(X(a(\lambda), t)\right) d\lambda + \int_{\mathbb{T}} u(X(a(\lambda), t), t) \cdot \partial_\lambda \left(u(X(a(\lambda), t), t)\right) d\lambda. \tag{1.75}$$

The second term on the right side of (1.75) equals

$$\frac{1}{2}\int_{\mathbb{T}} \partial_\lambda |u(X(a(\lambda), t), t)|^2 d\lambda = 0$$

since by assumption a is $\mathbb{T}$-periodic. On the other hand, using the momentum equation (1.27), and undoing the change of variables, we obtain that the first term on the right side of (1.75) equals

$$-\int_{\mathbb{T}} (\nabla p)(X(a(\lambda), t), t) \cdot \partial_\lambda \left(X(a(\lambda), t)\right) d\lambda = -\int_{C(t)} \nabla p \cdot ds = 0$$

since the curve $C(t)$ is closed. This concludes the proof of the theorem. □

Note that if $C = \partial S$, a surface in the fluid, the Stokes theorem yields

$$\Gamma_C = \int_C u \cdot ds = \int_S \omega \cdot n dA = \int_S \omega \cdot dA,$$

and this the circulation around C equals the flux of vorticity across a surface whose boundary is C. The Kelvin circulation theorem then yields:

Corollary 1.27. *The flux of vorticity across a surface moving with the fluid is constant in time.*

Definition 1.28 (Vortex line)**.** A vortex line L is a curve that is tangent at all its points to the vorticity vector.

Definition 1.29 (Vortex sheet)**.** A vortex sheet S is a surface that is tangent at all its points to the vorticity vector.

It then immediately follows from Theorem 1.26 that:

Corollary 1.30. *Assume that L and S are a vortex line, respectively a vortex sheet. Then their push forwards under the flow map of the Euler equations, $L(t) = X(L,t)$ and $S(t) = X(S,t)$, a are also a vortex line, respectively a vortex sheet.*

Exercise 1.31. Prove Corollary 1.30.

Definition 1.32 (Vortex tube)**.** A vortex tube is a collection of vortex lines, with the property that the vorticity is tangent to the side surface of the vortex tube (i.e. this side surface is a vortex sheet), and the vorticity is orthogonal to cross sections of the vortex tube.

Theorem 1.33 (Helmholtz theorem on vortex tubes)**.** *If C_1 and C_2 are two closed curves encircling the vortex tube, then $\Gamma_{C_1} = \Gamma_{C_2}$. This invariant is called the* strength of the vortex tube. *Moreover, the strength of the vortex tube is constant in time, as the tube is carried around the fluid along the Lagrangian paths.*

Exercise 1.34. Apply the divergence theorem in the "cylindrical" region of the vortex tube determined by the closed curves C_1, C_2, and the boundary of the vortex tube, in order to prove the first part of the Helmholtz theorem. The second part follows directly from the circulation theorem 1.26.

1.14 Globally conserved quantities

For both $d = 2$ and $d = 3$, and a smooth solution u of the Euler equations (1.27)–(1.28) the following quantities are constant in time:

$$\text{Mean velocity: } \int_\Omega u(x,t)dx$$
$$\text{Mean vorticity: } \int_\Omega \omega(x,t)dx$$

for $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, and

$$\text{Kinetic energy: } E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2 dx. \tag{1.76}$$

which holds also when Ω is a domain with boundary. The proof of (1.76) is seen by multiplying (1.29) by u_j and summing over $j \in \{1,\ldots,d\}$. Using (1.30) and integration by parts we obtain

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega u_j u_j dx &= -\int_\Omega u_j \partial_j p dx - \int_\Omega u_j u_i \partial_i u_j dx \\ &= -\int_\Omega u \cdot \nabla p dx - \frac{1}{2}\int_\Omega u \cdot \nabla(|u|^2) dx \\ &= -\int_{\partial\Omega} p u \cdot n dx - \frac{1}{2}\int |u|^2 u \cdot n dx = 0.\end{aligned}$$

It turns out that the above computation may be justified as soon as $u \in L^3_t C^{1/3+\delta}_x$ for some $\delta > 0$ (cf. [CET94]). The keyword here is the *Onsager conjecture*.

For $d = 3$, the following quantities are also constant in time:

$$\text{Helicity: } \int_{\mathbb{R}^3} u(x,t) \cdot \omega(x,t) dx \tag{1.77}$$
$$\text{Fluid impulse: } \frac{1}{2} \int_{\mathbb{R}^3} x \times \omega(x,t) dx$$
$$\text{Momentum of fluid impulse: } \frac{1}{3} \int_{\mathbb{R}^3} x \times (x \times \omega(x,t)) dx.$$

Exercise 1.35. Prove that the helicity (1.77) is conserved in 3D Euler.

For $d = 2$, the following quantities are also constant in time:

$$\text{Modified energy: } \int_{\Omega} \psi(x,t)\omega(x,t) dx \tag{1.78}$$
$$\text{Fluid impulse: } \frac{1}{2} \int_{\mathbb{R}^2} (x_2 - x_1)\omega(x,t) dx$$
$$\text{Momentum of fluid impulse: } -\frac{1}{3} \int_{\mathbb{R}^2} |x|^2 \omega(x,t) dx.$$

To see that (1.78) holds, we notice that upon integrating by parts,

$$\begin{aligned} \int_{\Omega} \psi(x,t)\omega(x,t) dx &= \int_{\Omega} \psi(x,t) \nabla^{\perp} \cdot u(x,t) dx \\ &= -\int_{\Omega} \nabla^{\perp}\psi(x,t) \cdot u(x,t) dx = -2E(t) \end{aligned}$$

since the stream function obeys a Dirichlet boundary condition on $\partial\Omega$.

In two dimensions we have in fact infinitely many conservation laws, which may also be coercive. More precisely

$$\text{Functions of vorticity, also known as Casimirs: } I_F = \int_{\Omega} F(\omega(x,t)) dx. \tag{1.79}$$

To see that (1.79) is conserved in time, we use the change of variables $x \mapsto X(a,t)$, which combined with (1.71) yields

$$I_F(t) = \int_{\Omega} F(\omega(X(a,t),t)) \det(\nabla_a X)(a,t) da = \int_{\Omega} F(\omega_0(a)) da = I_F(0)$$

where we have used that $X(\cdot,t) \colon \Omega \to \Omega$ is an invertible volume preserving diffeomorphism.

1.15 Special solutions

1.15.1 Steady solutions in 2D Euler

Given a smooth function F, let the stream function solve

$$-\Delta\psi = \omega = F(\psi).$$

The resulting velocity field $u = \nabla^{\perp}\psi$ is a steady solution on the 2D Euler equations, with associated constant (thus zero) pressure.

1.15.2 Shear flows in 2D Euler

Let $U : \mathbb{R} \to \mathbb{R}$ be a smooth function. Defining

$$u(x_1, x_2) = (U(x_2), 0)$$

it is not hard to see that this is a steady solution of the 2D Euler equations, called a *shear flow* with profile U. The associated pressure is again trivial.

1.15.3 Shear flows in 3D Euler

Let $f, g : \mathbb{R}$. Then the 2.5-dimensional flow

$$u(x_1, x_2, x_3, t) = (f(x_2), 0, g(x_1 - tf(x_2)))$$

is an exact solution of 3D Euler, with trivial associated pressure. Note that this shear flow is time-dependent.

Exercise 1.36. We shall show later on in the course that the Euler equations are well-posed in $C_x^{1,\alpha}(\mathbb{T}^3)$ when $\alpha \in (0, 1)$. Using the above shear flow, show that the Euler equations are ill-posed in $C_x^{\alpha}(\mathbb{T}^3)$ when $\alpha \in (0, 1)$. (See [BT10]).