A Ordinary Differential Equations

Proposition A.1 (Grönwall Inequality)**.** *Let $u(t)$ and $v(t)$ be non-negative, continuous functions on $[a, b]$. Assume there exists a constant $C > 0$ such that*

$$v(t) \leq C + \int_a^t v(s)u(s)ds$$

for all $a \leq t \leq b$. Then we have that

$$v(t) \leq C \exp\left(\int_a^t u(s)ds\right)$$

for all $a \leq t \leq b$. Thus, if $C = 0$, then $v(t) \equiv 0$.

Theorem A.2 (Picard-Lindelöf existence and uniqueness)**.** *Fix $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$, $a, b > 0$, and define the parallelepiped $R = \{(t, y) \colon t_0 \leq t \leq t_0 + a,\ |y - y_0| \leq b\}$. We consider the ordinary differential equation*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{A.1}$$

where f is continuous on R (with maximum equaling $M > 0$), and uniformly Lipschitz continuous w.r.t. y. Then (A.1) has a unique solution $y(t)$ defined on $[t_0, t_0 + T]$, where $T = \min\{a, b/M\}$.

Exercise A.3 (Example of Non-uniqueness)**.** Consider f as above that is not uniformly Lipschitz continuous. Show that uniqueness of solutions to (A.1) may fail.

Theorem A.4 (Osgood uniqueness condition)**.** *Assume that the function f in Theorem A.2 is not necessarily Lipschitz continuous, but instead we are given that it obeys*

$$\sup_{(t,x) \neq (t,y) \in R} \frac{|f(t, x) - f(t, y)|}{\omega(|x - y|)} \leq C$$

where the function $\omega \colon [0, 2b] \to [0, \infty)$ is continuous, $\omega(0) = 0$, $\omega(\cdot) > 0$ on $(0, 2b]$, and

$$\lim_{\delta \to 0+} \int_\delta^{2b} \frac{1}{\omega(r)} dr = +\infty.$$

Then the solution to (A.1) has a unique solution.

Theorem A.5 (Peano existence theorem)**.** *Fix $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$, $a, b > 0$, and define the parallelepiped $R = \{(t, y) \colon t_0 \leq t \leq t_0 + a,\ |y - y_0| \leq b\}$. Assuming that f is continuous on R (with maximum equaling $M > 0$), the (A.1) has at least one solution $y(t)$ defined on $[t_0, t_0 + T]$, where $T = \min\{a, b/M\}$.*

Exercise A.6 (Maximal time of existence)**.** Let $f(t, y)$ be continuous on an open set E in the (t, y) plane. By using the Peano theorem A.5 we know that the ordinary differential equation (A.1) has a solution $y(t)$ on some interval J. This interval is called maximal if we cannot extend $y(t)$ as a solution to (A.1) on any interval J_1 with $J_1 \supsetneq J$. Show that as $t \to \partial J$, we have that $y(t) \to \partial E$.

B Sobolev Spaces

We briefly review Sobolev spaces; see [Eva98, AF03] for more details. We mostly follow [Eva98], as the introduction there is clean and sufficient for what we need here. For $\alpha \in \mathbb{N}^d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, we use the multi-index notation:

$$\partial^\alpha f = \partial_{\alpha_1} \partial_{\alpha_2} \ldots \partial_{\alpha_d} f,$$

and denote $|\alpha| = \sum_{i=1}^d |\alpha_i|$. For any integer $0 \leq n < \infty$ and index $p \in [1, \infty]$, define the (inhomogeneous) Sobolev norm

$$\|f\|_{W^{n,p}} = \sum_{\alpha \in \mathbb{N}^d : |\alpha| \leq n} \|\partial^\alpha f\|_{L^p},$$

and the homogeneous Sobolev norm

$$\|f\|_{\dot{W}^{n,p}} = \sum_{\alpha \in \mathbb{N}^d : |\alpha| = n} \|\partial^\alpha f\|_{L^p}.$$

For $\Omega \subset \mathbb{R}^d$ or $\mathbb{T}^d$, we define the Sobolev space $W^{n,p}$ (or $\dot{W}^{n,p}$) as the closure of C_c^∞ (smooth, compactly supported functions) with respect to the norm $W^{n,p}$ (or $\dot{W}^{n,p}$). The L^2 based spaces are Hilbert spaces, denoted $H^n = W^{n,2}$ and $\dot{H}^n = \dot{W}^{n,2}$, equipped with the inner product

$$< f, g >_{H^s} = \sum_{\alpha \in \mathbb{N}^d : |\alpha| \leq n} \int \partial^\alpha f \partial^\alpha g dx$$
$$< f, g >_{\dot{H}^s} = \sum_{\alpha \in \mathbb{N}^d : |\alpha| = n} \int \partial^\alpha f \partial^\alpha g dx.$$

By definition, C_c^∞ is of course dense in Sobolev spaces, but these spaces can contain many rough functions. More or less equivalently, we may define Sobolev spaces as the locally integrable functions $f \in L^1_{loc}$ such that the distributional derivatives are in the requisite L^p spaces. This definition is equivalent up to minor details regarding decay at infinity (with the latter definition, $\dot{H}^1$ would contain $f(x) = 1$ whereas with the former, the one we are adopting, it would not). On a bounded domain, the definitions agree entirely.

One can have unbounded functions in Sobolev spaces:

Exercise B.1. Let $\{q_n\}_{n=1}^\infty \subset [0, 1]^2$ be an enumeration of the points with rational coordinates in the unit box. Let $\phi(x)$ be a smooth, non-negative function equal to 1 for $x \in [0, 1]^2$ and 0 for $|x| \geq 2$. Show that the function

$$f(x) = \sum_{n=1}^\infty 2^{-n} \log |x - q_n| \, \phi(x),$$

is in H^1, despite being unbounded on every open set in $[0, 1]$.

However, functions in Sobolev spaces cannot have anything like a "jump" (which at first glance seem more innocuous than the pathology above):

Exercise B.2. Show that $\mathbf{1}_{B(0,1)} \notin W^{n,p}(\mathbb{R}^d)$ for all $n \geq 1$, $p \in [1, \infty]$, and $d \geq 1$.

A very important property of Sobolev spaces is that under certain conditions, it is possible to define restrictions of functions in Sobolev spaces to some lower dimensional manifolds (in particular, the boundaries of domains), which is of course not the case for L^p functions. However, since we will not be dealing with this much in these notes, we refer the reader to [Eva98, AF03] for more details.

Very useful inequalities involve the relationship of one Sobolev space to another. For example, we have the following set of inequalities, which also serve as embedding theorems; see [Eva98] for a proof.

Theorem B.3 (Gagliardo-Nirenberg-Sobolev)**.** *The following two inequalities for all $d \geq 1$:*

- *Let $f \in \dot{W}^{1,p}(\mathbb{R}^d)$ and $1 \leq p < d$. Then,*

$$\|f\|_{L^{p^\star}} \lesssim_{p,d} \|f\|_{\dot{W}^{1,p}},$$

where

$$p^\star = \frac{dp}{d-p}.$$

- *Let $f \in W^{1,p}(U)$ for some bounded, open set $U \subset \mathbb{R}^d$ with a C^1 boundary. Then, for $1 \leq p < d$*

$$\|f\|_{L^{p^\star}} \lesssim_{p,d} \|f\|_{W^{1,p}}.$$

Remark B.4. You do not need to memorize $p^\star$: there is only one choice and you can derive it on the spot by considering functions of the type $f_\lambda(x) = f(\lambda x)$ for all λ.

For $p > d$ we can actually deduce some Hölder continuity; see [Eva98] for a proof.

Theorem B.5 (Morrey's inequality)**.** *Let $f \in W^{1,p}(\mathbb{R}^d)$ with $d < p \leq \infty$. Then,*

$$\|f\|_{C^{0,1-\frac{d}{p}}} \lesssim_{p,d} \|f\|_{W^{1,p}},$$

after possibly re-defining f on a set of measure zero. Suitably localized versions also hold (for example, on $\mathbb{T}^d$).

Remark B.6. Note that neither of the last two embedding theorems are valid at $p = d$. This is one of those irritating borderline cases where everything you want fails by a logarithm (recall Exercise B.1 shows that there exists functions $f \in H^1(\mathbb{R}^2)$ with $f \notin L^\infty$).

Theorem B.7 (Local Poincaré)**.** *Let $U \subset \mathbb{R}^d$ be bounded, connected, and open with a C^1 boundary. Then for all $u \in \dot{H}^1(U)$ there holds*

$$\left\| u - \frac{1}{|U|}\int_U u dx \right\|_{L^2(U)} \leq \frac{1}{\lambda_1^N} \|\nabla u\|_{L^2(U)},$$

where λ_1^N is the first non-zero *eigenvalue of $-\Delta$ on U with homogeneous Neumann boundary conditions on ∂U.*

Remark B.8. Theorem B.7 holds also on $\mathbb{T}^d$ and in fact on any compact manifold without boundary.

Theorem B.9 (Poincaré)**.** *Let $U \subset \mathbb{R}^d$ be bounded, connected, and open with a C^1 boundary. Then for all $u \in \dot{H}_0^1(U)$ (the closure of $C_0^\infty(U)$, the set of smooth functions which vanish on ∂U, under the $\dot{H}^1$ norm), there holds*

$$\|u\|_{L^2(U)} \leq \frac{1}{\lambda_1^D} \|\nabla u\|_{L^2(U)},$$

where λ_1^D is the first eigenvalue of $-\Delta$ on U with homogeneous Dirichlet boundary conditions on ∂U.

Remark B.10. Poincare's inequalities are equivalent to a spectral gap for the operator $-\Delta$ with associated boundary conditions. Any other elliptic operator with a spectral gap will satisfy something similar, including, for example, $-\Delta$ on $\mathbb{R}^d$ in weighted spaces (this also shows why Poincaré's inequalities fail without weights on $\mathbb{R}^d$).

C Compactness Theorems

The most fundamental compactness theorem is the Arzela-Ascoli theorem. This version of (i) is from [Theorem 4.43, [Fol13] with Exercise 61]. Part (ii) is a slight variant thereof – in fact (ii) follows from (i) (using the characterization of pre-compactness as total boundedness; see [Fol13] for more details).

Theorem C.1 (Arzela-Ascoli)**.** *The following holds:*

(i) Let X be a compact Hausdorff space and Y be a complete metric space. Supppose that $\mathcal{F} \subset C(X;Y)$ (continuous maps from X to Y):

  (a) $\mathcal{F}$ is uniformly totally bounded in the sense that there exists a totally bounded set $Z \subset Y$ such that $f(X) \subset Z$ for all $f \in \mathcal{F}$;

  (b) $\mathcal{F}$ is equicontinuous, in the sense that for all $x \in X$, $\epsilon > 0$ there exists a neighborhood U of x such that $d(f(x), f(x')) < \epsilon$ for all $x' \in U$ and all $f \in \mathcal{F}$.

  Then $\overline{\mathcal{F}}$ is a compact subset of $C(X,Y)$ in the topology induced by the uniform metric.

(ii) Let X be a Hausdorff space and Y a complete metric space. Suppose that

  (a) $\mathcal{F}$ is tight in the sense that there is a point $y_0 \in Y$ such that for all $\epsilon > 0$ there is a compact set K_ϵ such that $\sup_{f\in\mathcal{F}} d(f|_{X\setminus K_\epsilon}, y_0) < \epsilon$;

  (b) $\mathcal{F}$ is uniformly totally bounded in the sense that there exists a totally bounded set $Z \subset Y$ such that $f(X) \subset Z$ for all $f \in \mathcal{F}$;

  (c) $\mathcal{F}$ is equicontinuous, in the sense that for all $x \in X$, $\epsilon > 0$ there exists a neighborhood U of x such that $d(f(x), f(x')) < \epsilon$ for all $x' \in U$ and all $f \in \mathcal{F}$.

An important application of Theorem C.1 is the following compactness theorem for Sobolev spaces. See [Eva98] for a proof.

Theorem C.2 (Rellich embedding theorem)**.** *Let $U \subset \mathbb{R}^d$ be an open, bounded domain with a C^1 boundary. Then for all $p < d$ and $1 \le q < p^\star$, we have*

$$W^{1,p}(U) \subset\subset L^q(U).$$

As a consequence, bounded sequences in $W^{1,p}(U)$ have subsequences which converge in L^q for all $1 \le q < p^\star$.

Exercise C.3. Prove that Theorem C.2 is false for $q = p^\star$, although Theorem B.3 shows that $W^{1,p} \subset L^{p^\star}$ (Hint: recall how $p^\star$ was determined in the first place!).

Exercise C.4. Let $\{f_n\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^d)$ for $p < d$ be a uniformly bounded sequence. Prove that there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ and a function f such that such that $f_{n_k} \to f$ in $L^q(B(0,R))$ for all $1 \le q < p^\star$ and all $0 < R < \infty$. Show by example that it is not necessarily the case that $f_{n_k} \to f$ in $L^q(\mathbb{R}^d)$. What additional condition would be sufficient to deduce this?

Exercise C.5. Prove that Theorem C.2 is false if U is not bounded. In fact, prove that Theorem C.2 is false even modulo translations if U is unbounded (it suffices to prove there exists a sequence of $\{f_n(x\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^d)$ such that $\|f_n\|_{W^{1,p}}(\mathbb{R}^d) = 1$ but there does not exist any sequence $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^d$ such that $g_n(x) = f_n(x + y_n)$ has convergent subsequences in L^q for any $1 \le q < p^\star$).

In PDEs, there is an additional useful lemma which comes up naturally when dealing with evolution equations.

Theorem C.6 (Aubin-Lions compactness, C^0 version)**.** *Let $X \subset Y \subset Z$ be separable, reflexive Banach spaces, such that the embedding $X \subset Y$ is compact, and the embedding $Y \subset Z$ is continuous. Let $T > 0$, and assume that we have a sequence of functions $\{u_n\}_{n\geq 1}$ such that there is an $M > 0$,*

$$\|u_n(t)\|_X \leq M$$
$$\{u_n\} \text{ is uniformly equicontinuous on } [0,T] \text{ with values in } Z.$$

Then the sequence $\{u_n\}$ is pre compact in $C([0,T];Y)$.

Proof of Theorem C.6. First we need a Lemma (see e.g. [Sho13, CF88]) which allows to, in some sense, interpolate the Y norm between the Z and X norms.

Exercise C.7. Prove that for all $\delta > 0$, there exists a constant C_δ such that for all $x \in X$ there holds

$$\|x\|_Y \leq \delta \|x\|_X + C_\delta \|x\|_Z . \tag{C.1}$$

Using Exercise C.7 we can prove that $u_n \in C([0,T];Y)$ and is uniformly equicontinuous. Let $\epsilon > 0$ let $0 \leq t < s \leq T$, then, choosing $\delta = M^{-1}\epsilon$ in (C.1),

$$\begin{aligned} \|u_n(t) - u_n(s)\|_Y &\leq \delta \|u_n(t) - u_n(s)\|_X + C_\delta \|u_n(t) - u_n(s)\|_Z \\ &\leq 2\epsilon + C_\delta \|u_n(t) - u_n(s)\|_Z . \end{aligned}$$

By equicontinuity in Z there exists an $\eta > 0$ such that $|t - s| < \eta$ implies $C_\delta \|u_n(t) - u_n(s)\|_Z < \epsilon$ uniformly in n and hence it follows that $\{u_n\}_{n\geq 1}$ is equicontinuous in Y. The result now follows from Arzela-Ascoli. □

Theorem C.8 (Aubin-Lions compactness, L^p version)**.** *Let $X \subset Y \subset Z$ be separable, reflexive Banach spaces, such that the embedding $X \subset Y$ is compact and the embedding $Y \subset Z$ is continuous. Let $T > 0$, and assume that we have a sequence of functions $\{u_n\}_{n\geq 1}$ such that*

$$\{u_n\} \text{ is uniformly bounded in } L^p(0,T;X)$$
$$\{\partial_t u_n\} \text{ is uniformly bounded in } L^q(0,T;Z)$$

where $\infty > p, q > 1$. Then the sequence $\{u_n\}$ is (strongly) precompact in $L^p(0,T;Y)$.

Proof of Theorem C.8. The following proof is basically from [Lemma 8.4, [CF88]]. First, note that $u_n \in C^{1-\frac{1}{q}}([0,T];Z)$ uniformly in n by Hölder's inequality. By Banach-Alaoglu (and $1 < p < \infty$ so that $L^p(0,T;X)$ is separable and reflexive), it follows that there is a weakly convergent subsequence $u_{n_k} \rightharpoonup u$ in $L^p(0,T;X)$. Hence the sequence $v_k = u_{n_k} - u \rightharpoonup 0$. The result follows if we can verify that $v_k \to 0$ strongly in $L^p(0,T;Y)$. First, we note that for any interval $I \subset [0,T]$

$$\int_I v_k(\tau) d\tau \rightharpoonup 0 \quad \text{in} \quad X.$$

To see this, note that for any $\phi \in X^*$ we have $\mathbf{1}_{t\in I}\phi \in L^{\frac{p}{p-1}}(0,T;X^*)$ and hence

$$\langle \int_0^T \mathbf{1}_{t\in I} v_k(t) dt, \phi \rangle_{X\times X^*} = \int_0^T \langle v_k(t), \mathbf{1}_{t\in I}\phi \rangle_{X\times X^*} dt \to 0.$$

By the compact embedding, it follows that the convergence is strong in Y. Let $\epsilon > 0$ be fixed and let $\bar{v}_k^\epsilon(t)$ be the step function of time-averages of length ϵ. By the dominated convergence theorem it follows that

$\lim_{k\to 0}\|v_k^\epsilon\|_{L^r(0,T;Z)}$ for all $1 < r < \infty$ (noting that $\|v_k(t)\|_Z \lesssim 1$). By the uniform Hölder continuity in time in Z we have

$$\|v_k(t) - \bar{v}_k^\epsilon(t)\|_Z \lesssim \epsilon^{1-\frac{1}{q}},$$

and hence it follows that for all $\epsilon > 0$, we can find k sufficiently large such that

$$\int_0^T \|v_k(t)\|_Z^p\,dt \lesssim \int_0^T \|\bar{v}_k^\epsilon(t)\|_Z^p\,dt + \int_0^T \|v_k(t) - \bar{v}_k^\epsilon(t)\|_Z^p\,dt \lesssim \epsilon + \epsilon^{p-\frac{p}{q}}T.$$

Therefore, $\lim_{k\to\infty}\|v_k\|_{L^p(0,T;Z)} = 0$. By Exercise (C.7), we have for all $\delta > 0$,

$$\|v_k\|_{L^p(0,T;Y)} < \delta\,\|v_k\|_{L^p(0,T;X)} + C_\delta\,\|v_k\|_{L^p(0,T;Z)}$$

and so by choosing k sufficiently large, we can make

$$\|v_k\|_{L^p(0,T;Y)} < 2\delta,$$

from which it follows that $v_k \to 0$ strongly in $L^pI(0,T;Y)$. This completes the theorem. □

D Properties of mollifiers

Definition D.1 (Standard family of mollifiers)**.** Let $\phi\colon \mathbb{R}^d \to \mathbb{R}$ be C^∞ smooth non-negative radial function, which additionally is radially non-increasing, has finite moments $\int_{\mathbb{R}^d} |x|^m \phi(x)dx$ of any degree $m \geq 0$, and has mass 1, i.e. $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For any $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^d}\,\phi\left(\frac{x}{\varepsilon}\right).$$

The family of mass 1 non-negative functions $\{\phi_\varepsilon\}_{\varepsilon>0}$ is called a standard family of mollifiers. The corresponding mollification operator $\mathcal{J}_\varepsilon$, is defined by

$$\mathcal{J}_\varepsilon f(x) = (\phi_\varepsilon * f)(x)$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Remark D.2 (Fourier representation). Note that we have

$$\widehat{\mathcal{J}_\varepsilon f}(\xi) = \widehat{\phi}(\varepsilon\xi)\widehat{f}(\xi)$$

for any $\xi \in \mathbb{R}^d$, so that the mollifier $\mathcal{J}_\varepsilon$ is given by a Fourier multiplier at frequency scale $\xi \approx \varepsilon^{-1}$. Note that the function $\widehat{\phi}$ is also radial. The mass normalization of ϕ amounts to $\widehat{\phi}(0) = 1$, so that by the mean value theorem we have

$$\widehat{\phi}(\varepsilon\xi) = 1 + \varepsilon \int_0^1 (\partial_\xi \widehat{\phi})(\lambda\varepsilon\xi)d\lambda.$$

Since $\|\partial_\xi \widehat{\phi}(\xi)\|_{L^\infty} \leq \||x|\phi(x)\|_{L^1}$, it thus follows that $\widehat{\phi}(\varepsilon\xi) = 1 + \mathcal{O}(\varepsilon)$, from which it follows that $\mathcal{J}_\varepsilon$ yields an approximation of the identity in a suitable sense. Note moreover that $\mathcal{J}_\varepsilon$ being a Fourier multiplier with real-symbol it *commutes with derivatives*, and is *self-adjoint on* L^2.

Remark D.3 (Support of the mollifier). Sometimes it is necessary to have a mollifier of compact support. In this case the function ϕ is merely take to belong to $C_0^\infty(\mathbb{R}^d)$. Otherwise, a convenient function ϕ is given by the Gaussian: then $\widehat{\phi}$ is also a Gaussian, and moreover the mollifier $\mathcal{J}_\varepsilon$ yields real-analytic functions (which is not the case if ϕ had compact support).

Remark D.4 (Periodic mollifiers). In the case of a periodic domain, one may simply define a periodic mollifier $\mathcal{J}_\varepsilon$ as a Fourier (series) multiplier operator with the symbol $c\exp(-|k|^2)$, where c is a suitable normalization constant.

Proposition D.5 (Properties of standard mollifiers)**.** *Let $\mathcal{J}_\varepsilon$ be a family of standard mollification operators. The following properties hold true:*

(i) $\mathcal{J}_\varepsilon f \in C^\infty$*, for any* $f \in L^1_{\text{loc}}$

(ii) For $f \in L^p$*, we have* $\|\mathcal{J}_\varepsilon f\|_{L^p} \leq \|f\|_{L^p}$*. When* $1 \leq p < \infty$ *we have* $\|\mathcal{J}_\varepsilon f - f\|_{L^p} \to 0$ *as* $\varepsilon \to 0$*, while if* $f \in C_0$*, then* $\mathcal{J}_\varepsilon f \to f$ *uniformly on compact subsets of* $\mathbb{R}^d$*.*

(iii) For $f \in H^s$ *we have that* $\|\mathcal{J}_\varepsilon f - f\|_{H^{s-\delta}} = \mathrm{o}(\varepsilon^\delta)$ *as* $\varepsilon \to 0$*.*

(iv) For $f \in L^2$ *we have that* $\|\mathcal{J}_\varepsilon f\|_{H^s} \lesssim \varepsilon^{-s}\|f\|_{L^2}$*, and* $\|\mathcal{J}_\varepsilon f\|_{L^\infty} \lesssim \varepsilon^{-d/2}\|f\|_{L^2}$*.*

E Fourier analysis

Recall the definition of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (see e.g. [Fol13]). For $f(x) \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transform $\mathcal{F}[f](\xi) = \widehat{f}(\xi)$

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$

We define the inverse Fourier transform via

$$\mathcal{F}^{-1}[\widehat{f}](x) = f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

Both transforms are well defined on L^1. By duality, the Fourier transform and its inverse can be extended to L^2 (and the inverse transform really does invert the Fourier transform) [Fol13]. With these conventions, we have the following:

$$\begin{aligned} \int f(x)\overline{g}(x)dx &= \int \widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi \\ \widehat{fg} &= \frac{1}{(2\pi)^{d/2}} \widehat{f} * \widehat{g} \\ (\widehat{\nabla f})(\xi) &= i\xi\widehat{f}(\xi). \end{aligned}$$

For $m \in L^1_{loc}$, we define the Fourier multiplier $m(\nabla)f$ via (whether or not its defined for a given f)

$$(\widehat{m(\nabla)f})(\xi) = m(i\xi)\widehat{f}(\xi).$$

Sobolev spaces based on L^2 can then be defined for all $s \in \mathbb{R}$ via

$$\begin{aligned} \|f\|_{H^s} &= \|\langle\nabla\rangle^s f\|_{L^2} \\ \|f\|_{\dot{H}^s} &= \||\nabla|^s f\|_{L^2}. \end{aligned}$$

Using the inverse Fourier transform we get the following from Cauchy-Schwarz with $s > d/2$:

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^s}. \tag{E.1}$$

By density of $\mathcal{S} \subset H^s$, this implies further that if $f \in H^s$ for $s > d/2$ then $f \in C^0$ (as usual, up to re-definition on a set of measure zero).

Exercise E.1. Show that (E.1) does not hold for $d = 2$ and $s = 1$ (it fails for all d and $s = d/2$, but an example is easiest to justify in $\mathbb{R}^2$).

F Singular integrals

Recall that the Newtonian potential is, in 3D for example,

$$\mathcal{N}(x) = \frac{1}{4\pi |x|}.$$

Two derivatives of $\mathcal{N}$ gives

$$\partial_{ij}\mathcal{N}(x) = -\frac{1}{4\pi |x|^3}\left(\delta_{ij} - 3\frac{x_i x_j}{|x|^2}\right).$$

Normally we would want

$$\partial_{ij}\left(\mathcal{N} * u\right) = \left(\partial_{ij}\mathcal{N}\right) * u,$$

BUT its not so easy, since $\partial_{ij}\mathcal{N}$ is not locally integrable, so it doesn't work out quite like this and its not clear we can make sense of $\partial_{ij}\mathcal{N}$ even in the sense of distributions. Instead, we recall the notion of principle value [Ste93]:

$$\partial_{ij}\left(\mathcal{N} * u\right) := PV \int \partial_{ij}\mathcal{N}(x-y)u(y)dy = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \partial_{ij}\mathcal{N}(x-y)u(y)dy.$$

Recall that with this definition we get the following in the sense of distributions at least [Eva98]

$$\Delta(\mathcal{N} * u) = u.$$

For the other derivatives we get a non-local singular integral. However, we have a variety of methods to get estimates on them, usually broadly referred to as Calderón-Zygmund theory or singular integral theory (see [LR02, Ste93]). For example, there is this one [Ste93]:

Theorem F.1. *Let $K \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be such that for all $\alpha \in \mathbb{N}^d$ and $0 < a < b$,*

$$\int_{a \le |x| \le b} K(x)dx = 0$$
$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha K(x)\right| \lesssim_\alpha |x|^{-d-|\alpha|}.$$

Then, define the operator

$$Tf = PV \int K(x-y)f(y)dy = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} K(x-y)f(y)dy.$$

This operator is well-defined as a bounded operator $T : L^p \to L^p$ for $1 < p < \infty$,

$$\|Tf\|_{L^p} \lesssim_p \|f\|_p.$$

In particular, this holds for standard convolution type operators, known as Calderón-Zygmund operators *of the form*

$$Tf(x) = p.v. \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)dy \tag{F.1}$$

where the function $\Omega \colon \mathbb{S}^{d-1} \to \mathbb{R}$ is smooth and obeys the cancellation property

$$\int_{\mathbb{S}^{d-1}} \Omega(y)dy = 0.$$

Remark F.2. Theorem F.1 is in general false at $p = 1$ and $p = \infty$. However, we do have (see [Ste93])

$$\|Tf\|_{BMO} \lesssim \|f\|_{BMO}.$$

Remark F.3. For getting L^p estimates using the Green's functions on bounded domains we need still more generality. We need to consider operators of the kind $\int_\Omega G(x, y)u(y)dy$. Such generalizations are possible [Ste93].

Another convenient variant is to determine which Fourier multipliers can be extended to operators on L^p (Fourier multipliers are easiest to understand in L^2 due to Plancherel's theorem). For Fourier multipliers (like $\partial^\alpha \mathcal{N}$ on $\mathbb{R}^d$ or $\mathbb{T}^d$) we can use the following theorem.

Theorem F.4 (Marcinkiewicz multiplier theorem)**.** *Let m be homogeneous of degree zero and C^∞ away of the origin such that*

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha m(\xi)\right| \lesssim_\alpha |\xi|^{-|\alpha|},$$

holds for all α. Then $m(\nabla)f$ defines a bounded operator on L^p for all $1 < p < \infty$ and

$$\|m(\nabla)f\|_{L^p} \lesssim_p \|f\|_{L^p}.$$

Moreover,

$$\|m(\nabla)f\|_{BMO} \lesssim \|f\|_{BMO}.$$

The first appplication of Theorem F.4 is deducing boundedness of the Riesz transforms:

Corollary F.5. *Let $R_i(\nabla) = i\xi_i |\xi|^{-1}$. Then R_i defines a bounded multiplier on L^p and BMO:*

$$\|R_i f\|_{L^p} \lesssim \|f\|_{L^p}$$
$$\|R_i f\|_{BMO} \lesssim \|f\|_{BMO}.$$

From Corollary F.5, one can deduce similar boundedness of the derivatives of solutions to $-\Delta\phi = f$ (on $\mathbb{R}^d$ and $\mathbb{T}^d$). Indeed,

$$\widehat{\partial_{ij}\mathcal{N}}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2},$$

so

$$\partial_{ij}\Delta^{-1} f = R_i R_j f.$$

Hence, we deduce

Corollary F.6. *Let $f \in L^p$. Then for $1 < p < \infty$,*

$$\|\partial_{ij}(\mathcal{N} * f)\|_{L^p} \lesssim_p \|f\|_{L^p}. \tag{F.2}$$

Moreover,

$$\|\partial_{ij}(\mathcal{N} * f)\|_{BMO} \lesssim \|f\|_{BMO} \lesssim \|f\|_{L^\infty}. \tag{F.3}$$

Further, if $f \in L^p$ for $p \in (d, \infty)$, then if we write $u = \nabla\mathcal{N} * f$, we have

$$\|u\|_{C^{0,1-\frac{d}{p}}} \lesssim \|f\|_{L^p}. \tag{F.4}$$

Finally, there also holds

$$\sup_{x,y:|x-y|<1/2} \frac{|u(x) - u(y)|}{|x-y| \log|x-y|^{-1}} \lesssim \|f\|_{L^\infty} + \|f\|_{L^1}. \tag{F.5}$$

Remark F.7. Notice that $\Delta\phi$ is obviously in L^∞ if $f \in L^\infty$, but it is *not* true that the full set of derivatives $\partial_{ij}\phi \in L^\infty$ if $f \in L^\infty$. Instead, we only get $\partial_{ij}\phi \in BMO$.

Proof. The first part follows from Corollary F.5. Inequality (F.4) can be proved directly, but alternatively is derived from (F.2) and Morrey's inequality [Eva98]. Inequality (F.5) is most easily proved directly. However, there does exist an extension with L^∞ replaced with BMO, which is significantly harder to prove; this can be deduced from (F.3) and a suitable endpoint variant of Morrey's inequality that shows $\nabla u \in BMO$ along with a uniform integrability requirement (L^1 suffices) implies log-Lipschitz regularity (see [Theorem A.2, Proposition A.3 [McM14]] or [AB15]). Let us demonstrate the proof of (F.5) directly using elementary techniques – we follow the proof of [Lemma 8.1, [MB02]]. Define $\epsilon = |x-y| < 1/2$ and split the integral into three disjoint pieces:

$$\begin{aligned}
|u(x) - u(y)| &\lesssim \left| \int \left(\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right) f(z) dz \right| \\
&\leq \left| \int_{B(x,2\epsilon)} \left(\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right) f(z) dz \right| \\
&\quad + \left| \int_{B(x,2)\setminus B(x,2\epsilon)} \left(\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right) f(z) dz \right| \\
&\quad + \left| \int_{\mathbb{R}^d\setminus B(x,2)} \left(\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right) f(z) dz \right| \\
&= T_1 + T_2 + T_3.
\end{aligned}$$

For T_1 we can use (recall $\epsilon = |x-y|$),

$$T_1 \lesssim \|f\|_{L^\infty} \int_{B(x,2\epsilon)} \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}} \right) dz \lesssim \|f\|_{L^\infty} |x-y|.$$

By the mean value theorem,

$$\left| \frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right| \lesssim \left(\sup_{\theta\in[0,1]} \frac{1}{|x-z+\theta(x-y)|^d} \right) |x-y|. \tag{F.6}$$

For T_3, we note that the denominator in (F.6) is uniformly bounded below by an $O(1)$ constant on the support of the integrand, and hence

$$T_3 \lesssim \|f\|_{L^1} |x-y|.$$

It is in the intermediate region, T_2, where we lose the logarithm. On the support of the integrand in T_2, note that the denominator in (F.6) is uniformly bounded below by $|x-z|$ (because $B(x,2)\setminus B(x,2\epsilon)$ and $\epsilon = |x-y|$). Hence,

$$T_2 \lesssim \|f\|_{L^\infty} |x-y| \int_{B(x,2)\setminus B(x,2\epsilon)} \frac{1}{|x-z|^d} dz \lesssim \|f\|_{L^\infty} |x-y| \left(\log |x-y|^{-1} \right),$$

which completes the proof of (F.5). □

G Littlewood-Paley decompositions and some basic appplications

The fact that the Fourier transform on $\mathbb{T}^d$ is discrete can be convenient; it can often be even more convenient if you group frequencies dyadically. We can do something similar in $\mathbb{R}^d$ by using a dyadic frequency decomposition known as the Littlewood-Paley decomposition.

We will only present the homogeneous Littlewood-Paley decomposition, which means that no particular length-scale is chosen a priori as "long" or "short" – in the inhomogeneous decomposition, frequencies $\lesssim 1$ are treated differently. For scale-invariant problems, it makes sense to use the homogeneous Littlewood-Paley decomposition instead, however, there are plenty of uses for the inhomogeneous decomposition and other more unusual variants as well.

Let $\psi \in C_0^\infty(\mathbb{R}_+; \mathbb{R}_+)$ be such that $\psi(\xi) = 1$ for $\xi \leq 1/2$ and $\psi(\xi) = 0$ for $\xi \geq 3/4$ and define $\rho(\xi) = \psi(\xi/2) - \psi(\xi)$, supported in the range $\xi \in (1/2, 3/2)$. Then we have the partition of unity for $\xi > 0$,

$$1 = \sum_{M \in 2^{\mathbb{Z}}} \rho(M^{-1}\xi),$$

where we mean that the sum runs over the dyadic integers $M = ..., 2^{-j}, ..., 1/4, 1/2, 1, 2, 4, ..., 2^j, ...$ and we define the cut-off $\rho_M(\xi) = \rho(M^{-1}\xi)$, each supported in $M/2 \leq \xi \leq 3M/2$. For $f \in L^2(\mathbb{R}^d)$ we define

$$f_M = \rho_M(|\nabla|)f, \qquad f_{<M} = \sum_{K \in 2^{\mathbb{Z}}: K < M} f_K.$$

Usually f_M is called a *frequency shell* or the $M - th$ *dyadic shell* or something to this effect. We make use of the notation

$$f_{\sim M} = \sum_{K \in 2^{\mathbb{Z}}: \frac{1}{C}M \leq K \leq CM} f_K,$$

for some constant C which is independent of M. Generally the exact value of C which is being used is not important; what is important is that it is finite and independent of M. The dyadic shells are not quite orthogonal in L^2, but they satisfy a number of properties which makes them almost like an orthogonal decomposition (it will be important that ρ is a smooth function, which means we cannot do a piecewise disjoint frequency decomposition, which would be orthogonal in L^2 but would not satisfy good properties in L^p for $p \neq 2$). What we can say about the almost orthogonality and approximate projection property is the following:

Lemma G.1. *Let $f \in L^2(\mathbb{R}^d)$. Then,*

$$\lim_{j \to \infty} \left\| f - \sum_{2^{-j} < M < 2^j} f_M \right\|_2 = 0.$$

Moreover, there holds the almost orthogonality and the approximate projection property:

$$\|f\|_2^2 \approx \sum_{M \in 2^{\mathbb{Z}}} \|f_M\|_2^2 \tag{G.1a}$$

$$\|f_M\|_2 \approx \|(f_M)_{\sim M}\|_2 . \tag{G.1b}$$

More generally, if $f = \sum_j D_j$ for any D_j with $\frac{1}{C}2^j \subset \operatorname{supp} D_j \subset C2^j$ it follows that

$$\|f\|_2^2 \approx_C \sum_{j \in \mathbb{Z}} \|D_j\|_2^2 . \tag{G.2}$$

Next note that by definition,

$$f_N = (f_N)_{N/2\leq\cdot\leq 2N}. \tag{G.3}$$

The next set of inequalities expand the estimates into L^p regularity and begin to show the usefulness of the Littlewood-Paley decomposition.

Theorem G.2 (Bernstein's inequalities)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$ (and extended via density to more general classes), there holds for all $1 \leq p \leq q \leq \infty$ and $s \in \mathbb{R}$:*

$$\|f_N\|_{L^p} \lesssim_{p,d} \|f\|_{L^p} \tag{G.4a}$$
$$\|f_{\sim N}\|_{L^p} \lesssim_{p,d} \|f\|_{L^p} \tag{G.4b}$$
$$\||\nabla|^s f_N\|_{L^p} \approx_{p,s,d} N^s \|f_N\|_{L^p} \tag{G.4c}$$
$$\||\nabla|^s f_{\sim N}\|_{L^p} \approx_{p,s,d} N^s \|f_{\sim N}\|_{L^p} \tag{G.4d}$$
$$\|f_N\|_{L^q} \lesssim_{p,q,d} N^{\frac{d}{p}-\frac{d}{q}} \|f_N\|_{L^p} \tag{G.4e}$$
$$\|f_{\sim N}\|_{L^q} \lesssim_{p,q,d} N^{\frac{d}{p}-\frac{d}{q}} \|f_{\sim N}\|_{L^p}. \tag{G.4f}$$

Remark G.3. Note that $p \leq q$ and hence (G.4e) and (G.4f) only go one way.

Proof. If we denote $\phi_N = \mathcal{F}^{-1}[\rho_N]$ then we have

$$f_N = (2\pi)^{d/2}\phi_N * f.$$

Next we note that $\phi_N(x) = N^d\phi_1(Nx)$ and that $\phi_1 \in \mathcal{S}$. Hence, by Young's convolution inequality, (G.4a) follows. The proof of (G.4b) is similar.

For (G.4c), note first that $|\xi|^s \rho_N(\xi) = N^s \left(\left|N^{-1}\xi\right|^s \rho(N^{-1}\xi)\right)$. It follows that $\phi_N^s(x) = \mathcal{F}^{-1}[|\cdot|^s \rho_N] = N^s N^d \phi_1^s(Nx)$. Moreover, $\phi_1^s \in \mathcal{S}$. Therefore it follows from Young's convolution inequlity that

$$\||\nabla|^s f_N\|_{L^p} \lesssim_p N^s \|f\|_{L^p}.$$

By (G.3) and a similar argument we deduce

$$\||\nabla|^s f_N\|_{L^p} = \left\||\nabla|^s (f_N)_{N/2\leq\cdot\leq 2N}\right\|_{L^p} \lesssim_p N^s \|f_N\|_{L^p}.$$

The reverse inequality in (G.4c) follows from the fact that we can take $s \in \mathbb{R}$. Indeed,

$$N^s \|f_N\|_{L^p} = N^s \left\||\nabla|^{-s} |\nabla|^s f_N\right\|_{L^p} \lesssim \||\nabla|^s f_N\|_{L^p}.$$

Inequality (G.4d) follows similarly.

Consider (G.4e). As above, $\phi_N(x) = N^d\phi_1(Nx)$ and that $\phi_1 \in \mathcal{S}$. Therefore it follows by Young's convolution inequality that

$$\begin{aligned}\|f_N\|_{L^p} &\lesssim \|\phi_N\|_{L^{1+\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^q} \\ &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|f\|_{L^q}.\end{aligned}$$

However, this implies

$$\|f_N\|_{L^p} \lesssim \|(f_N)_{\sim N}\|_{L^p} \lesssim N^{\frac{d}{p}-\frac{d}{q}} \|f_N\|_{L^q}.$$

□

The proof of the last inequality is beyond the scope of the course. See e.g. [Ste93] for a proof.

Theorem G.4 (Littlewood-Paley square function inequality)**.** *For all $1 < p < \infty$, there holds*

$$\|f\|_{L^p} \approx_p \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |f_N|^2 \right)^{1/2} \right\|_{L^p} .$$

One of the great things about Littlewood-Paley decompositions is that it allows very easy proofs of a lot of inequalities involving Sobolev spaces which otherwise might be much harder. In fact, its arguably *too* powerful, since once one learns it, one often forgets how to use the less elegant "physical-side" Sobolev space tools, which while much uglier, are necessary if one wants to easily localize or consider Sobolev spaces on bounded domains.

The first application we will show is the Sobolev embedding theorem.

Proposition G.5 (Sobolev embedding)**.** *For all $s \in (0, \frac{d}{2})$, there exists a constant $C_{SE} = C_{SE}(s, d)$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$, there holds for $\frac{d}{q} = \frac{d}{2} - s$:*

$$\|f\|_{L^q} \leq C_{SE} \|f\|_{\dot{H}^s} .$$

Remark G.6. This embedding is not compact, even locally – check this by considering the sequence of functions $f_\lambda = \lambda^{d/p} f(x\lambda)$, for some compactly supported f. The inequality above does not see the scale λ, and so one can send $\lambda \to 0$ and one gets a sequence which is not compact in L^q.

Proof. By the square function inequality and the triangle inequality (using that $q > 2$),

$$\|f\|_{L^q} \approx \left\| \sum_{N \in 2^{\mathbb{Z}}} |f_N|^2 \right\|_{L^{q/2}}^{1/2} \lesssim \left(\sum_{N \in 2^{\mathbb{Z}}} \|f_N\|_{L^q}^2 \right)^{1/2} .$$

By Bernstein's inequality we have for $s = \frac{d}{q} - \frac{d}{2}$,

$$\|f_N\|_{L^q} \lesssim N^{\frac{d}{q} - \frac{d}{2}} \|f_N\|_{L^2} \approx \| |\nabla|^s f_N \|_{L^2} .$$

Therefore,

$$\left(\sum_{N \in 2^{\mathbb{Z}}} \|f_N\|_{L^q}^2 \right)^{1/2} \lesssim \left(\sum_{N \in 2^{\mathbb{Z}}} \| |\nabla|^s f_N \|_{L^2}^2 \right)^{1/2} \approx \|f\|_{\dot{H}^s} .$$

□

As a straightforward application of Littlewood-Paley theory, we prove the following inequality, useful in the proof of local well-posedness in $\dot{H}^{1/2}(\mathbb{T}^3)$.

Proposition G.7. *Let $f, g : D \to \mathbb{C}^n$ where $D = \mathbb{R}^3$ or $\mathbb{T}^3$. Then*

$$\|fg\|_{\dot{H}^{1/2}} \lesssim \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^1} . \tag{G.5}$$

Proof. We will just prove the theorem in the case of $\mathbb{R}^3$; the $\mathbb{T}^3$ case is similar (but easier since Littlewood-Paley theory is strictly speaking not necessary) Note

$$\|fg\|_{\dot{H}^{1/2}}^2 \approx \sum_{N\in 2^{\mathbb{Z}}} N \|(fg)_N\|_2^2 = \sum_{N\in 2^{\mathbb{Z}}} N \left\|(f_{<N/8}g)_N\right\|_2^2 + \sum_{N\in 2^{\mathbb{Z}}}\sum_{M>N/8} N \|(f_M g)_N\|_2^2$$
$$= T_1 + T_2.$$

Turn to T_1 first. By the definition of the Littlewood-Paley projections,

$$T_1 \lesssim \sum_{N\in 2^{\mathbb{Z}}} N \left\|f_{<N/8}g_{\sim N}\right\|_2^2.$$

Then, by Hölder's inequality, Bernstein's inequalities, and Sobolev embedding we have

$$T_1 \lesssim \sum_{N\in 2^{\mathbb{Z}}} N \left\|f_{<N/8}g_{\sim N}\right\|_2^2 \lesssim \sum_{N\in 2^{\mathbb{Z}}} N \left\|f_{<N/8}\right\|_6^2 \|g_{\sim N}\|_3^2 \lesssim \sum_{N\in 2^{\mathbb{Z}}} N^2 \left\|f_{<N/8}\right\|_{\dot{H}^1}^2 \|g_{\sim N}\|_2^2$$
$$\lesssim \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^1},$$

where the last line followed from almost orthogonality. Turn next to T_2, which we treat similarly with Sobolev embedding:

$$T_2 \lesssim \sum_{N\in 2^{\mathbb{Z}}}\sum_{M>N/8} N \|f_M\|_3^2 \|g\|_6^2 \lesssim \sum_{N\in 2^{\mathbb{Z}}}\sum_{M>N/8} \frac{N}{M} M^2 \|f_M\|_2^2 \|g\|_{\dot{H}^1}^2 \lesssim \sum_{M\in 2^{\mathbb{Z}}}\sum_{N<8M} \frac{N}{M} M^2 \|f_M\|_2^2 \|g\|_{\dot{H}^1}^2$$
$$\lesssim \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^1}.$$

□

A similar application of Littlewood-Paley theory is the classical Sobolev product rule, which we can now verify for fractional derivatives. The proof is a variation of the proof of Proposition G.7, and is left as an exercise.

Proposition G.8 (Product rule)**.** *Let $f, g \in H^s$ for $s > 0$. Then there holds*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{L^\infty} + \|g\|_{H^s} \|f\|_{L^\infty}.$$

From (E.1)*, this implies that H^s for $s > d/2$ is a Banach algebra:*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}.$$

Proof. Exercise. □

In fluid mechanics, the notion of a Besov space comes up every so often. These spaces are a slight generalization of Sobolev spaces which can be natural or useful when dealing with certain critical or borderline situations. We will just give the definition and we will not dwell on them here:

$$\|f\|_{B^s_{p,q}} = \left(\|f_{<1}\|_{L^p}^q + \sum_{N\in 2^{\mathbb{Z}}: N\geq 1} N^{sq} \|f_N\|_{L^p}^q\right)^{1/q},$$
$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{N\in 2^{\mathbb{Z}}} N^{sq} \|f_N\|_{L^p}^q\right)^{1/q}.$$

The normal Sobolev spaces are a subset:

$$\|f\|_{\dot{H}^s} \approx \|f\|_{\dot{B}^s_{2,2}}.$$

H Basic properties of Poisson's equation and the heat equation

We recall a few basic concepts from introductory PDEs, following mostly [Eva98].

H.1 Poisson's equation

Recall that in $\mathbb{R}^d$, $d \geq 2$, the fundamental solution to Poisson's equations are

$$\mathcal{N}(x) = \frac{1}{2\pi} \log|x-y| \quad \text{for} d = 2$$
$$\mathcal{N}(x) = c_d \, |x-y|^{2-d} \quad \text{for} d \geq 3,$$

for suitable constant c_d (it is multiple of the surface area of the unit sphere). From there we can solve Poisson's equation

$$-\Delta u = f.$$

Theorem H.1 (Poisson's equation in 2D)**.** *Let $d = 2$ and suppose $f \in L^1 \cap L^p$ for $p \in (2, \infty)$ such that $\int f dx = 0$ and $\int |fx| \, dx < \infty$. Then*

$$u(x) = \mathcal{N} * f$$

is a weak solution to

$$-\Delta u = f$$

which is unique in the class $L^\infty \cap \dot{W}^{2,p}$.

Remark H.2*.* The mean zero and finite first moment assumptions amount additional, more stringent, requirements on the low frequencies of f, this is important for deducing the L^∞. For proofs in this more general class, see e.g. [GT01].

In $d \geq 3$ we do not need the extra requirements on low frequencies (we can also drop the L^1 assumption depending on how carefully we replace it).

Theorem H.3 (Poisson's equation in $d \geq 3$)**.** *Let $d \geq 3$ and suppose $f \in L^1 \cap L^p$ for $p \in (d, \infty)$. Then*

$$u(x) = \mathcal{N} * f$$

is a weak solution to

$$-\Delta u = f$$

which is unique in the class $L^\infty \cap \dot{W}^{2,p}$.

H.2 Heat equation

Consider solutions to the Cauchy problem on $\mathbb{R}^d$ (for now)

$$\partial_t u - \nu \Delta u = 0$$
$$u(0, x) = f(x).$$

We know from basic PDEs [Eva98] that, at least for a *very* wide class of initial data, the unique solution is given by the following

$$u(t,x) = G_\nu(t,\cdot) * f,$$

where the fundamental solution is given by

$$G_\nu(t,x) = \frac{1}{(4\pi\nu t)^{d/2}} e^{-\frac{|x|^2}{4\nu t}}. \tag{H.1}$$

The fundamental solution is just the solution to the IVP with initial data given by the Dirac δ measure. We will also use the "linear propagator" notation

$$u(t,x) = e^{t\nu\Delta} f.$$

The operator $e^{t\nu\Delta}$ formally generates a continuous semi-group over a wide class of initial data, which we will sometimes refer to as the *heat propagator*. To fix ideas, let us consider initial data in L^p for some $p \in [1,\infty]$.

From (H.1) one can easily prove the following a priori estimates on the heat propagator, which include regularization, decay, and gain of integrability (sometimes called hypercontractivity).

Proposition H.4. *The following a priori estimate holds for $p,q,s,r \in \mathbb{R}$ which satisfy $p \geq q \geq 1$ and $r \geq s \geq 0$:*

$$\left\| e^{\nu t\Delta} f \right\|_{\dot{W}^{r,p}} \lesssim_{r,s} (\nu t)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{(r-s)}{2}} \|f\|_{\dot{W}^{s,q}}.$$

Corollary H.5. *If $f \in L^p$ For any L^p with $p \in [1,\infty]$ then $e^{t\nu\Delta} f \in C^\infty$ for all $t > 0$.*

Proof. Let us assume $r,s \in \mathbb{N}$. In this case, the result follows from Young's (convolution) inequality; indeed for $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$ and $\beta \in \mathbb{N}^d$ with $\beta \leq \alpha$ and $|\beta| = s$:

$$\left\| D^\alpha e^{\nu t\Delta} f \right\|_{L^p} \lesssim \left\| D^{\alpha-\beta} G_\nu(t,\cdot) \right\|_{L^r} \left\| D^\beta f \right\|_{L^q},$$

for r satisfying

$$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}.$$

The proposition then follows by direct computation from (H.1). For the more general case of $r,s \in \mathbb{R}_+$, one can use Littlewood-Paley theory on $\mathbb{R}^d$. $\square$

Finally, we note that the heat propagator on $\mathbb{R}^d$ (or $\mathbb{T}^d$) has a convenient form in Fourier analysis:

$$\widehat{u}(t,\xi) = e^{-t\nu|\xi|^2} \widehat{f}(\xi).$$

From this formula it is immediate to deduce instanteous real analytic regularity.

On the torus or on a bounded domain it is easy to prove exponential decay of the heat equation in L^2. This is due to the spectral gap of the operator $-\Delta$; it fails on $\mathbb{R}^d$ because on the unbounded domain the spectrum is continuous and goes all the way to zero – so no gap.

Proposition H.6. *Let $f \in L^2(\mathbb{T}^d)$ then,*

$$\left\| e^{t\nu\Delta} f - \langle f \rangle_x \right\|_{L^2} \lesssim e^{-\lambda_1 \nu t} \|f\|_{L^2},$$

where λ_1 is the first non-zero eigenvalue of the Laplacian.

Proof. Denote $u(t) = e^{\nu t \Delta} f$. This is done via the Poincare inequality:

$$\frac{1}{2}\frac{d}{dt}\left\|u(t) - \langle u(t)\rangle_x\right\|_{L^2}^2 = -\nu\left\|\nabla u\right\|_{L^2}^2 \le -\nu\lambda_1\left\|u - \langle u\rangle_x\right\|_{L^2}^2 .$$

□

For a general study of semigroup theory and how it is related spectral theory for a wider class of operators, see [EN00].

I Fixed point theorems

One of the simplest and most important fixed point theorems is usually called the *contraction mapping principle*; it admits a very simple proof and is applicable in many PDE settings.

Theorem I.1 (Contraction mapping principle)**.** *Let X be a Banach space, $B \subset X$ be a closed subset, and $\Phi : B \to B$ be Lipschitz continuous on B in the norm topology with Lipschitz constant $L < 1$. Then there exists a unique fixed point $x \in B$ satisfying $\Phi(x) = x$.*

Proof. The proof follows by Picard iteration. Let $x_0 \in B$ be arbitrary. Then, for $n \geq 1$ recursively define

$$x_n = \Phi(x_{n-1}).$$

It is clear that the sequence $\{x_n\}_{n=0}^{\infty} \subset B$. We will next prove that this sequence is Cauchy, which suffices by the completeness of X and the fact that B is closed. First, we claim that there exists a $C > 0$ and $\delta = \log_2 L^{-1} > 0$ such that for all $1 \leq n < \infty$,

$$\|x_n - x_{n-1}\| \leq C 2^{-\delta n}. \tag{I.1}$$

First, consider the case $n = 1$:

$$\|\Phi(x_0) - x_0\| \leq \|\Phi(x_0)\| + \|x_0\|,$$

which suffices for C chosen sufficiently large. Next, assume (I.1) holds for all $k \leq n - 1$. Then,

$$\|x_n - x_{n-1}\| = \|\Phi(x_{n-1}) - \Phi(x_{n-2})\| \leq L \|x_{n-1} - x_{n-2}\| \leq CL2^{-\delta(n-1)} \leq L2^{\delta}C2^{-\delta n}.$$

Therefore (using $\delta = \log_2 L^{-1}$), the induction step follows and (I.1) holds for all n. Hence, for all $0 \leq n < m < \infty$ we get

$$\|x_n - x_m\| \leq \sum_{j=n}^{m-1} \|x_j - x_{j+1}\| \lesssim \sum_{j=n}^{m-1} 2^{-\delta j} \lesssim_{\delta} 2^{-\delta n},$$

and hence the sequence is Cauchy and the theorem is proved. □

The contraction mapping principle gives us an easy tool for doing abstract nonlinear perturbation, as seen in the following theorem, which is not as general as we can make it, but serves to demonstrate what is possible. The 4 is also not optimal and the restriction to nonlinearities which are approximately quadratic is also not important, more general powers work too.

Theorem I.2. *Let X be a Banach space. Suppose that $\mathcal{N}(x) : X \to X$ satisfies for some C_1, C_2,*

$$\|\mathcal{N}(x)\| \leq C_1 \|x\|^2$$
$$\|\mathcal{N}(x) - \mathcal{N}(y)\| \leq C_2 (\|x\| + \|y\|) \|x - y\|.$$

Then for all $x_0 \in X$ with $\|x_0\| < (4C_1)^{-1}$, there exists a unique solution to the nonlinear equation

$$x = x_0 + \mathcal{N}(x)$$

and the dependence of x on x_0 is Lipschitz continuous in the norm topology.

Proof. Define the mapping $\Phi[y]$ with

$$\Phi[y] = x_0 + \mathcal{N}(y).$$

Fix $\epsilon > 0$ to be specified shortly. First we claim that for ϵ sufficiently small, $\Phi : \overline{B(0,\epsilon)} \to \overline{B(0,\epsilon)}$. By our assumptions we have,

$$\|\Phi[y]\| \leq \|x_0\| + C_1 \|y\|^2 .$$

For $\epsilon < \frac{1}{2C_1}$ and $\|x_0\| < \frac{\epsilon}{2}$ the claim is verified. For Lipschitz continuity we have

$$\|\Phi[y_1] - \Phi[y_2]\| \leq C_2 \left(\|y_1\| + \|y_2\|\right) \|y_1 - y_2\| ,$$

and hence for $\epsilon < \frac{1}{2C_2}$, the mapping Φ is Lipschitz continuous with Lipschitz constant strictly less than one. It follows from Theorem I.1 that there is a unique solution $x \in \overline{B(0,\epsilon)}$ with $\Phi[x] = x$, that is,

$$x = x_0 + \mathcal{N}(x).$$

Notice that we chose $\|x_0\| < \frac{1}{4C_1}$. Lastly, we want to verify that the solution map is Lipschitz continuous with respect to x_0. Let $M[x_0] \mapsto x$ and suppose $M[x_0] = y_0$ and $M[x_1] = y_1$. Then,

$$\begin{aligned} \|y_1 - y_2\| &\leq \|x_0 - x_1\| + \|\mathcal{N}(y_1) - \mathcal{N}(y_2)\| \\ &\leq C_2 \left(\|y_1\| + \|y_2\|\right) \|y_1 - y_2\| . \end{aligned}$$

Then if $\|y_1\| + \|y_2\| \leq 2\epsilon < \frac{1}{C_2}$, then

$$\|y_1 - y_2\| \leq \frac{1}{1 - 2C_2\epsilon} \|x_0 - x_1\| ,$$

and the theorem is proved by choosing $\|x_0\| < \frac{1}{4C_1}$ and $\epsilon < \min\left(\frac{1}{2C_1}, \frac{1}{2C_2}\right)$. $\square$