# Exercise Sheet 1

1. Show that
$$\sum_{n\le x}\mu(n)\left\lfloor\frac{x}{n}\right\rfloor = 1.$$
Conclude that
$$\left|\sum_{n\le x}\frac{\mu(n)}{n}\right| \le 1.$$

2. Show that
$$\sum_{d|(m,n)} d\mu\Big(\frac{n}{d}\Big) = \frac{\mu\big(\frac{n}{(m,n)}\big)\phi(n)}{\phi\Big(\frac{n}{(m,n)}\Big)}.$$

3. Show that
$$\sum_{n=1}^{\infty}\phi(n)\frac{x^n}{1-x^n} = \frac{x}{(1-x)^2}.$$

4. Show that
$$\sum_{n\le x}\frac{\phi(n)}{n} = \frac{6}{\pi^2}x + O(\log x).$$

5. Let $\mathbb{F}_q$ be a finite field with $q$ elements, $\psi$ an additive character and $\chi$ a multiplicative character of $\mathbb{F}_q$. The Gauss sum associated to $\psi$ and $\chi$ is defined to be
$$\tau(\chi,\psi) = \sum_{x\in\mathbb{F}_q^\times}\chi(x)\psi(x).$$
Show that if $\psi$ and $\chi$ are non-trivial, then
$$|\tau(\chi,\psi)| = \sqrt{q}.$$
**Hint:** Consider $|\tau(\chi,\psi)|^2$ and expand.

6. Let $\mathbb{F}_q$ be a finite field with $q$ elements, and let $\chi$, $\phi$ be multiplicative characters of $\mathbb{F}_q$. The Jacobi sum associated to $\chi$ and $\phi$ is given by
$$J(\chi,\phi) = \sum_{x\in\mathbb{F}_q}\chi(x)\phi(1-x) = \sum_{\substack{x,y\in\mathbb{F}_q\\ x+y=1}}\chi(x)\phi(y).$$

(a) Show that if $\chi$, $\phi$ and $\chi\phi$ are all non-trivial, then for any non-trivial additive character $\psi$ of $\mathbb{F}_q$ we have

$$J(\chi, \phi) = \frac{\tau(\chi, \psi)\tau(\phi, \psi)}{\tau(\chi\phi, \psi)}.$$

**Hint:** Consider $J(\chi, \phi)\tau(\chi\phi, \psi)$ and expand.

(b) Conclude that $|J(\chi, \phi)| = \sqrt{q}$.

7. Let $\mathbb{F}_q$ be a finite field with $q$ elements, with $q$ odd. Let $\chi_2$ denote the unique non-trivial multiplicative character of order 2 of $\mathbb{F}_q^\times$, and let $\psi$, $\eta$ be additive characters of $\mathbb{F}_q$. The associated Salié sum is defined by

$$T(\psi, \eta) = \sum_{x \in \mathbb{F}_q^\times} \chi_2(x)\psi(x)\eta(x^{-1}).$$

(a) Show that if $\psi$ and $\eta$ are non-trivial then

$$T(\psi, \eta) = \tau(\chi_2, \psi) \sum_{y^2 = 4a} \psi(y),$$

where $a \in \mathbb{F}_q^\times$ is such that $\eta(x) = \psi(ax)$ for all $x \in \mathbb{F}_q$.

(b) Conclude that $|T(\psi, \eta)| \leq 2\sqrt{q}$.

8. Use a contour integral involving the $\zeta$-function to compute

$$\sum_{n \leq x} 1.$$

Use the result of this computation to give a heuristic explanation for the value of $\zeta(0)$.

9. Find an asymptotic for

$$\sum_{n \leq x} \frac{\mu(n)^2}{\varphi(n)}.$$

10. Prove

$$|(s-1)\zeta(s)| \ll_\epsilon e^{|s|^{1+\epsilon}}.$$

Conclude that there exist $a, b$ such that

$$(s-1)\zeta(s) = e^{a+bs} s \prod_{0 < \text{Re}\rho < 1} (1 - \frac{s}{\rho}) e^{s/\rho} \prod_{n \geq 1} (1 + \frac{s}{2n}) e^{-s/2n}.$$

11. Use the argument principle to compute $N(T)$ as accurately as you can.

12. Using the identity

$$H_n = \int_0^1 1 + x + \cdots + x^{n-1} dx,$$

show that

$$\int_\varepsilon^\infty \frac{e^{-t}}{t} dt = \log \frac{1}{\varepsilon} - \gamma + O(\varepsilon).$$