Exercise Sheet 2

1. Suppose that we wish to find an upper bound on the number of integers between x and $x+y$ having no prime factors p with $y^{1/3} < p \leq y^{1/2}$.

   (a) Use Selberg's upper bound sieve to show that the number of such integers is at most $\frac{y}{1+\log(3/2)} \approx 0.71y$ plus an error term, and compute the error term.

   (b) Use Brun's combinatorial upper bound sieve to show that the number of such integers is at most $(1 - \log(3/2) + \frac{\log(3/2)^2}{2})y \approx 0.68y$ plus an error term, and compute the error term.

   (c) Compute an asymptotic for the number of such integers in the case $x = 0$.

2. Let R, κ be given positive integers. Suppose that for every prime p with $y^{\frac{1}{R+1}} < p \leq y^{\frac{1}{R}}$ we are given a set of "bad" congruence classes C_p (modulo p) of size κ. Let v be defined by

$$v = \sum_{y^{\frac{1}{R+1}} < p \leq y^{\frac{1}{R}}} \frac{\kappa}{p}.$$

   Suppose that we wish to find an upper bound on the number of integers n between x and $x+y$ such that for each prime p between $y^{\frac{1}{R+1}}$ and $y^{\frac{1}{R}}$, n is not congruent to an element of C_p (modulo p).

   (a) Let λ_k, $k = 0, ..., R$ be a collection of real numbers such that $\lambda_0 = 1$ and such that for every $m \geq 0$ we have

$$\sum_{k=0}^{R} \lambda_k \binom{m}{k} \geq 0.$$

   Show that the number of integers n as above is at most

$$y \sum_{k=0}^{R} \lambda_k \frac{v^k}{k!}$$

   plus an error term.

   (b) Given a collection of real numbers λ_k, $k = 0, ..., R$, define a polynomial $\theta(m)$ by

$$\theta(m) = \sum_{k=0}^{R} \lambda_k \binom{m}{k}.$$

Show that

$$\sum_{k=0}^{R} \lambda_k \frac{v^k}{k!} = e^{-v} \sum_{m=0}^{\infty} \theta(m) \frac{v^k}{k!}.$$

In more realistic sieving problems, the analogue of the right hand side of the above equation is often more convenient to work with than the analogue of the left hand side.

(c) Compute the polynomial $\theta(m)$ corresponding to Selberg's upper sieve, and compute the corresponding upper bound.

(d) Compute the polynomial $\theta(m)$ corresponding to Brun's combinatorial upper bound sieve, and compute the corresponding upper bound.

(e) Show that when $v \gg R \gg 1$, the upper bound from Selberg's sieve is smaller than the upper bound from Brun's combinatorial sieve.

3. Let $\rho(u)$ be the Dickman function, satisfying $\rho(u) = 1$ for $0 \leq u \leq 1$ and $u\rho'(u) = -\rho(u-1)$. Show that

$$\int_{u=0}^{\infty} \rho(u) du = e^{\gamma}.$$

4. Evaluate the inverse Mellin transform of $\zeta(s)\Gamma(s)$ in two different ways and compute in this way ζ at the negative integers.

5. Let χ be a primitive character modulo q. Prove that

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{\tau(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}\Big(\frac{n}{q}\Big)\overline{\chi}(n)$$

where $\tau(\chi)$ denotes the Gauss sum.

6. Find an asymptotic for

$$\sum_{n \leq x} \frac{\sigma(n)}{\varphi(n)} \mu(n)^2.$$

7. Find an asymptotic for

- $\sum_{n \leq x} d_k(n)$, where $k \in \mathbb{Z}$;
- $\sum_{n \leq x} d_\pi(n)$;
- $\sum_{n \leq x} d_i(n)$.

8. Show that

$$\sum_{\substack{n \leq R^u \\ p|n \Rightarrow p \leq R}} \frac{1}{n} = \prod_{p \leq R} \Big(1 - \frac{1}{p}\Big)^{-1} (1 + O(u^{1-u})).$$