Exercise Sheet 3

1. Let $f, g, h\colon \mathbb{N}^+ \longrightarrow \mathbb{U}$ be multiplicative functions and $x, y \in \mathbb{R}$. Recall the definition

$$\mathbb{D}(f,g;y,x)^2 = \sum_{y<p\leq x} \frac{1-\Re(f(p)\overline{g(p)})}{p}.$$

Check that $\mathbb{D}$ satisfies the triangle inequality

$$\mathbb{D}(f,h;y,x) \leq \mathbb{D}(f,g;y,x) + \mathbb{D}(g,h;y,x).$$

2. Recall from the discussion of the Selberg sieve that the main term of the upper bound to $\pi(x+y)-\pi(x)$ of the sieve was $\frac{y}{L}$ where

$$L = \sum_{\substack{r\leq z \\ p\mid r\Rightarrow p\leq R}} \frac{\mu(r)^2}{\varphi(r)}$$

where $z=\sqrt{y}$ and $R \leq z$ is a fixed power of z. Also, the sieve weights were given by

$$\lambda_r = \frac{\mu(r)r}{\varphi(r)L} \sum_{\substack{d\leq \frac{z}{r} \\ p\mid d\Rightarrow p\leq R \\ (d,r)=1}} \frac{\mu(d)^2}{\varphi(d)}.$$

Estimate λ_r in the case that $z=R^t$ with $t>1$.

3. Use Selberg's sieve to prove that

$$\pi(x;q,a) \leq \frac{(2+o(1))x}{\varphi(q)\log(x/q)}.$$

4. (a) For $\Im(\tau)>0$, define $\theta(\tau)=\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$. Show that $\theta(-1/\tau)=(-i\tau)^{\frac{1}{2}}\theta(\tau)$.
   (b) Use the previous exercise and the Mellin transform to prove the functional equation for ζ.