Exercise Sheet 4

1. The goal of this exercise is to explain the heuristics behind the so called "squareroot cancellation philosophy". Consider a sum of the form

$$\sum_{n=1}^{N} e(\theta_n)$$

where $\theta_n \in \mathbb{R}$ or $[0,1]$ for all $n \in \mathbb{N}$. The "squareroot cancellation philosophy" tells us that if the θ_n "vary enough" or behave "randomly enough" one expects that

$$\sum_{n=1}^{N} e(\theta_n) \ll \sqrt{N}. \tag{1}$$

An example of such sums are for example the Kloosterman sums. However, in many applications in which one expects that (1) holds, there is little hope that one can give an actual proof using currently available methods.

To explain why one expects (1), let (Ω, Σ, P) be a probability space and let

$$\theta_n \colon \Omega \longrightarrow [0,1]$$

be independently and uniformly distributed random variables, i.e., the probability distribution μ_{θ_n} of θ_n is given by the Lebesgue measure on $[0,1]$. This corresponds to the assumption that the θ_n behave "randomly enough". Now we consider the complex valued random variables

$$X_n = e(\theta_n) \colon \Omega \longrightarrow S^1 \subset \mathbb{C}.$$

(a) We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the canonical way. Compute the mean vector

$$\mu = \big(\mathbb{E}[\mathrm{Re}(X_n)], \mathbb{E}[\mathrm{Im}(X_n)]\big)$$

and the covariance matrix

$$\Sigma = \begin{pmatrix} \mathrm{Cov}\big(\mathrm{Re}(X_n), \mathrm{Re}(X_n)\big) & \mathrm{Cov}\big(\mathrm{Re}(X_n), \mathrm{Im}(X_n)\big) \\ \mathrm{Cov}\big(\mathrm{Re}(X_n), \mathrm{Im}(X_n)\big) & \mathrm{Cov}\big(\mathrm{Im}(X_n), \mathrm{Im}(X_n)\big) \end{pmatrix}$$

of X_n.

(b) Show that as $N \to \infty$,

$$\sqrt{N}\Big(\frac{1}{N}\sum_{n=1}^{N} X_n - \mu\Big) \Rightarrow \mathcal{N}_2(0, \Sigma),$$

where $\Rightarrow$ means convergence in distribution and $\mathcal{N}_2(0,\Sigma)$ denotes the multivariate normal distribution. Here we consider

$$\sum_{n=1}^{N} X_n \in \mathbb{C} = \mathbb{R}^2$$

and μ and Σ are as computed in a).

(c) By b) it seems reasonable that, for N big enough, we regard

$$\sqrt{N}\Big(\frac{1}{N}\sum_{n=1}^{N} e(\theta_n) - \mu\Big)$$

as being $\mathcal{N}_2(0,\Sigma)$ distributed, μ and Σ as in part a). Assuming this, what is the probability of

$$\Big|\sum_{n=1}^{N} e(\theta_n)\Big| \leq R\sqrt{N}?$$

Why does this suggest that

$$\sum_{n=1}^{N} e(\theta_n) \ll N^{\frac{1}{2}+\epsilon}$$

should be true?

2. Consider the equations

$$sQ'(s) + aQ(s) + bQ(s-1) = 0, \quad s > \beta$$

and

$$(sq(s))' = aq(s) + bq(s+1), \quad s > 0$$

(a) Show that

$$\langle Q, q\rangle := sQ(s)q(s) - b\int_{s-1}^{s} Q(x)q(x+1)dx$$

is constant.

(b) Show that if $a + b = n + 1 \in \mathbb{N}^+$, then $q(s)$ is a polynomial of degree n.

3. Show that

$$\rho(u) \leq \frac{e^{O(u)}}{(u\log u)^u}$$

where ρ denotes the Dickman function.

4. For $f \in C^1\big(\big[-\frac{1}{2},\frac{1}{2}\big]\big)$, check that

$$f(0) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)| + |f'(t)|dt.$$

5. Suppose that for every prime $p \le y^{1/u}$ we have a set of $\omega(p)$ "bad" congruence classes (modulo p), with $\omega(p) = \kappa$ for some fixed κ and all but finitely many primes p. Use Selberg's sieve to show that the number of integers n with $x < n \le x + y$ which are not in any of the bad congruences modulo any of the primes below $y^{1/u}$ is at most

$$\frac{y}{\sigma_\kappa(u)} \prod_{p \le y^{1/u}} \left(1 - \frac{\omega(p)}{p}\right) + \text{error},$$

where σ_κ solves the differential-delay equation

$$\begin{aligned} u^{-\kappa}\sigma_\kappa(u) &= \frac{1}{(2e^\gamma)^\kappa \Gamma(\kappa+1)} && 0 < u \le 2 \\ (u^{-\kappa}\sigma_\kappa(u))' &= -\kappa u^{-\kappa-1}\sigma_\kappa(u-2) && u \ge 2. \end{aligned}$$

6. Are there infinitely many integers n such that

$$\pi(n+100) - \pi(n) \ge 24?$$