Exercise Sheet 5

1. Show that for every q,
$$\sum_{\substack{1\le a<q\\(a,q)=1}} e\Big(\frac{a}{q}\Big) = \mu(q).$$

2. Use the large sieve to prove that
$$\pi(x;q,a) \le \frac{\big(2+o(1)\big)x}{\varphi(q)\log(x/q)}$$
where $q < x$.

3. Let χ be a non-trivial character modulo q. Show that
$$|L(1,\chi)| < 3\log q.$$

4. Show that for a prime p,
$$S(1,1;p^2) = 2p \text{ Re } e\Big(\frac{2}{p^2}\Big),$$
where $S(1,1;p^2)$ denotes the Kloosterman sum
$$S(1,1;p^2) = \sum_{\substack{0\le x\le p^2\\(x,p^2)=1}} e\Big(\frac{x+\overline{x}}{p^2}\Big).$$

5. Let $S \subset \{1,\dots,N\}$ such that for all primes p the number of residue classes of S modulo p is at most $\frac{p+1}{2}$; show that $|S| \ll \sqrt{N}$ and compute the implied constant.

6. Let $s \ge 1$ be a constant and $y = z^s$. Define
$$\pi^+(y,z) := \sum_{\substack{n\le y\\p|n\Rightarrow p\ge z}} 1-\lambda(n)$$
$$\pi^-(y,z) := \sum_{\substack{n\le y\\p|n\Rightarrow p\ge z}} 1+\lambda(n),$$
where $\lambda(n) = (-1)^{\Omega(n)}$ and $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity.

(a) Show that

$$\pi^+(y,z) = \pi^+(y,w) - \sum_{w \le p < z} \pi^-\Big(\frac{y}{p}, p\Big),$$
$$\pi^-(y,z) = \pi^-(y,w) - \sum_{w \le p < z} \pi^+\Big(\frac{y}{p}, p\Big).$$

(b) Show that

$$\pi^+(y,z) = F(s)\frac{y}{e^\gamma \log(z)} + 2H(s)\frac{y}{\log(z)^2} + O_s\Big(\frac{y}{\log(z)^3}\Big),$$
$$\pi^-(y,z) = f(s)\frac{y}{e^\gamma \log(z)} - 2h(s)\frac{y}{\log(z)^2} + O_s\Big(\frac{y}{\log(z)^3}\Big),$$

where

$$F(s) = \frac{2e^\gamma}{s} \qquad 1 \le s \le 3$$
$$\frac{d}{ds}(sF(s)) = f(s-1) \qquad s \ge 3$$
$$f(s) = \frac{2e^\gamma \log(s-1)}{s} \qquad 2 \le s \le 4$$
$$\frac{d}{ds}(sf(s)) = F(s-1) \qquad s \ge 2$$
$$H(s) = \frac{1}{s^2} \qquad 1 \le s \le 3$$
$$\frac{d}{ds}(s^2H(s)) = -sh(s-1) \qquad s \ge 3$$
$$h(s) = \frac{1}{s^2}\left(1 + \frac{1}{s-1} - \log(s-1)\right) \qquad 2 \le s \le 4$$
$$\frac{d}{ds}(s^2h(s)) = -sH(s-1) \qquad s \ge 2.$$

(c) Show that

$$F(s) - 1 = O(s^{-s})$$
$$1 - f(s) = O(s^{-s}).$$