Exercise Sheet 6

1. Let $k \geq 2$ be an integer. Show that $\theta = \frac{1}{k}$ is a strong exponent of distribution for τ_k.

2. Use Bombieri-Vinogradov to find an asymptotic for
$$\sum_{p \leq x} \tau(p-1).$$

3. Let $p \geq 3$ be a prime number and a an integer such that $(a,p) = 1$. For $M \geq 1$ an integer, show that
$$\left| \#\{m, n \leq M | mn \equiv a \pmod{p}\} - \frac{(p-1)M^2}{p^2} \right| < 4(\log p)^2 \sqrt{p}.$$

4. [a)]

   Show that
$$\limsup_{t \to \infty} |\zeta(1+it)| = \infty.$$

   (b) Give a heuristic for the growth of
$$\max_{T \leq t \leq 2T} |\zeta(1+it)|$$
   using the square-root cancellation philosophy.

5. (a) Show that
$$1 - \sigma_\kappa(s) = O(s^{-\frac{s}{2}})$$
   where σ_κ solves the differential-delay equation
$$\begin{aligned} s^{-\kappa}\sigma_\kappa(s) &= \frac{1}{(2e^\gamma)^\kappa \Gamma(\kappa+1)} && 0 < s \leq 2 \\ (s^{-\kappa}\sigma_\kappa(s))' &= -\kappa s^{-\kappa-1}\sigma_\kappa(s-2) && s \geq 2. \end{aligned}$$
   (Hint: Try using the result in Exercise Sheet 4, Problem 2.)

   (b) Check that if $\kappa = 1$, then we have
$$\sigma_1(s) = e^{-\gamma} \int_0^{\frac{s}{2}} \rho(u) du.$$