Exercise Sheet 8

1. Fill in the gaps in the proof of the Heath-Brown identity:

   Let $J \geq 1$, $X \geq 1$, $1 \leq n \leq 2X$. Then

$$\Lambda(n) = -\sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{m_1,\dots,m_j \leq X^{\frac{1}{J}}} \mu(m_1)\dots\mu(m_j) \sum_{\substack{n_1,\dots,n_j \\ m_1\cdots m_j n_1 \cdots n_j = n}} \log n_1.$$

2. Show that for $A \in \mathbb{N}$, $\alpha \in \mathbb{R}/\mathbb{Z}$, and $x \geq 2$,

$$\sum_{n \leq x} \mu(n) e(n\alpha) \ll_A \frac{x}{(\log x)^A}.$$

3. Consider the following random "stick breaking" process. Start with a stick of length 1, break off a piece of length X_1 uniformly distributed between 0 and 1, then break off a piece of the remaining stick of length X_2 uniformly distributed between 0 and $1 - X_1$, and so on: at the nth stage of the process we break off a piece of the remainder of the stick having length X_n uniformly distributed between 0 and $1 - X_1 - \cdots - X_{n-1}$. Finally, we define the random set $S = \{X_1, X_2, ...\}$.

   (a) Let X be a random element of the set S, chosen according to the distribution $\mathbb{P}[X = X_i] = X_i$. Show that X is uniformly distributed in $[0, 1]$.

   (b) Show that the probability that the largest element of S is at most $1/u$ is equal to $\rho(u)$. Compute the expected size of the largest element of S.

   (c) Let N be large, and let n be a random number between N and $2N$. Write $n = p_1 \cdots p_k$ with each p_i prime, and choose an index $1 \leq i \leq k$ at random, choosing i with probability $\frac{\log(p_i)}{\log(n)}$. Let $X = \frac{\log(p_i)}{\log(n)}$. Show that as N goes to infinity, the random variable X becomes uniformly distributed in $[0, 1]$. Conclude that the the random set $\{\frac{\log(p_1)}{\log(n)}, ..., \frac{\log(p_k)}{\log(n)}\}$ behaves like S (in some sense) as N goes to infinity.

   (d) Let n be large, and let σ be a uniformly random permutation from S_n. Let l be the length of the cycle of σ which contains 1, and let $X = \frac{l}{n}$. Show that $\mathbb{P}[l = i] = \frac{1}{n}$ for any integer $1 \leq i \leq n$. Conclude that when n goes to infinity, X becomes uniformly distributed in $[0, 1]$, and that the set of cycle lengths of σ divided by n behaves like S (in some sense).

   (e) Mimic part (c) to give an approximate description of the set of the degrees of the irreducible factors of a random polynomial of large degree in $\mathbb{F}_q[t]$.

4. Try to prove the following lemma of Zhang:

Let $n \geq 1$, $\frac{1}{10} < \sigma < \frac{1}{2}$, $t_1, \ldots, t_n$ with $\sum_i t_i = 1$. Then one of the following holds:

(i) One of the t_i is $\geq \frac{1}{2} + \sigma$.

(ii) There exists a partition $S \cup T = \{1, \ldots, n\}$ with $S \cap T = \emptyset$ such that

$$\frac{1}{2} - \sigma \leq \sum_{i \in S} t_i \leq \sum_{i \in T} t_i \leq \frac{1}{2} + \sigma.$$

(iii) There exist i, j, k distinct with

$$2\sigma \leq t_i \leq t_j \leq t_k \leq \frac{1}{2} - \sigma$$

and

$$t_i + t_j, t_i + t_k, t_j + t_k \geq \frac{1}{2} + \sigma.$$