2 The Navier-Stokes equations

Naturally, no real fluids are mathematically ideal and instead all have viscosity, which is the internal friction within the fluid. For general continuum materials, the forces on an element of material is of two types: "body forces", which act over a volume such as gravity, and "traction" forces, which arise due to internal forces between layers of the material and hence act on the boundaries of the material element. There is a theorem, *Cauchy's theorem*, which shows that by conservation of momentum, there is a matrix $\sigma(x)$, called the *Cauchy stress tensor*, such that the traction is given by $\sigma(x)n(x)$, the outward unit normal to the boundary of the element (see e.g. [MH94] for a proof of this and further discussions of continuum mechanics in general). Hence, the momentum balance on a given fluid element $W(t)$ is given by

$$\frac{d}{dt}\left(\int_{W(t)} \rho u(t,x)dx\right) = \left(\int_{\partial W(t)} \sigma n(x)dx\right).$$

The conservation of angular momentum can be used to further deduce that σ must be symmetric [MH94]. In the Euler equations, the stress tensor was assumed to be a scalar times the identity: $\sigma(x) = p(x)I$ so that all forces were normal to the boundary $\partial W(t)$ and isotropic (that is, the material properties are invariant under rotations). This means that in the Euler equations, layers of fluid can slide across one another without experiencing any force. For contrast, friction is a force which arises when two materials are sheared across one another, for example, a book sliding along a table, so viscosity will introduce forces when two layers of fluid slide against one another. Hence, we will need to update the stress tensor with a "shear stress" contribution. Let us write the tensor as a sum of viscous contributions and pressure contributions:

$$\sigma = pI + \sigma_{visc}.$$

In Newtonian fluid mechanics, we assume further that σ_{visc} depends linearly on ∇u and that σ_{visc} is invariant under rotations (isotropic). From these assumptions, one can prove that there exists constants λ, ν such that

$$\sigma_{visc} = \lambda(\text{div}u) + \nu\left(\nabla u + \nabla u^T\right);$$

see [CM93] for a proof. Due to incompressibility, the first term disappears, which leads us to

$$\sigma_{visc} = \nu\left(\nabla u + \nabla u^T\right).$$

The parameter $\nu > 0$ is the *dynamic viscosity* and is a material property of the fluid. From the units of σ, we can deduce that $[\nu] = ML^{-1}T^{-1}$. Re-deriving the momentum balance from the stress tensor gives in component form, the *incompressible Navier-Stokes equations*

$$\partial_t u_i + u_j\partial_j u_i = -\partial_i p + \nu\partial_j\left(\partial_j u_i + \partial_i u_j\right) \tag{2.1a}$$

$$= -\partial_i p + \nu\partial_{jj}u_i, \tag{2.1b}$$

where the last line followed via incompressibility. In vector form, the Navier-Stokes equations are then (adding back the equation for density)

$$\rho_t + u\cdot\nabla\rho = 0$$
$$\rho\left(\partial_t u + u\cdot\nabla u\right) = -\nabla p + \nu\Delta u$$
$$\nabla\cdot u = 0.$$

As above, we will be concentrating on the constant density case, $\rho(t,x) \equiv \rho_0$. In this case, we may divide the momentum equation through by ρ_0, and we get

$$\partial_t u + u\cdot\nabla u = -\frac{1}{\rho_0}\nabla p + \frac{\nu}{\rho_0}\Delta u.$$

The constant ρ_0 does not have an effect on the pressure, so up to re-definition we have

$$\partial_t u + u \cdot \nabla u = -\nabla p + \frac{\nu}{\rho_0}\Delta u.$$

The quantity $\nu\rho_0^{-1}$ is called the *kinematic viscosity*.

2.1 Non-dimensionalization

The fact that $\nu\rho_0^{-1}$ has units is in some sense not very nice: an equation, nor the physics it is meant to approximate, obviously cannot fundamentally depend on the units we are using to measure it. Naturally, there must be a way to "non-dimensionalize" the equations into a form which is unit-free. To do so, we choose natural scales U, L and M such that (being intentionally vague) in units of L, the main features are $O(1)$ in size (e.g. the domain, the characteristic length-scale of the flow features etc), in units of U the velocity is $O(1)$. That is, we re-scale (note the time scale $T = LU^{-1}$):

$$u^\star(t, x) = \frac{1}{U} u\left(\frac{tU}{L}, \frac{x}{L}\right)$$
$$p^\star(t, x) = \frac{1}{U^2} p\left(\frac{tU}{L}, \frac{x}{L}\right)$$

Note that U and L are only defined up to a vague multiplicative constant. If we are considering a domain with boundaries $\Omega \subset \mathbb{R}^d$, then it too is rescaled with λ. This gives

$$\partial_t u^\star + u^\star \cdot \nabla u^\star = -\nabla p^\star + \frac{\nu}{\rho_0 U L}\Delta u^\star$$
$$\nabla \cdot u = 0.$$

We will drop the $\star$'s and further assume we are always working in non-dimensionalized units. The dimensionless number in front of $\Delta u^\star$ is called the (inverse) *Reynolds number*:

$$\mathbf{Re} = \frac{\rho_0 U L}{\nu}.$$

Roughly speaking, one can see $\mathbf{Re}$ as a ratio between the influence of momentum and viscosity: at high $\mathbf{Re}$ the momentum is dominant and fluid will behave more like an ideal fluid, at low $\mathbf{Re}$, the fluid will be dominated by viscosity (though due to incompressibility it will not behave exactly like the heat equation). The behavior of an incompressible fluid depends only on this number, which is why a cup of water behaves different than a bathtub full of water, despite being both water. This can be convenient for experiments. For example, if you want to simulate a process involving water on a 100m scale, then instead of building a 100m water tank, you can build a 1m tank and fill it with something which has a kinematic viscosity 100 times lower than water (also easier said than done, but that's the basic idea). It is worth emphasizing again that $\mathbf{Re}$ is defined only up to some multiplicative constants. For example: when studying flow around disk, there is no one there to tell you whether to use the radius or the diameter or the circumference of the disk as L (for this particular example, it is convention to take the diameter). Hence, as a general rule, if someone just says "the laminar flow becomes unstable at Reynolds number 120" without telling you any details about the conventions they are taking, then they haven't exactly told you much, although in practice, usually everyone's conventions are equivalent up a factor of 2 or 3. Finally, in actual applications, it sometimes depends on the kinds of questions you are asking (and what your training is in!) to determine whether or not to classify a flow as "high Reynolds number" or "low Reynolds number".

The limit $\mathbf{Re} \to \infty$ is called the *inviscid limit* or *high Reynolds number limit* and *formally* the limit is the Euler equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p \\ \nabla \cdot u &= 0. \end{aligned}$$

Notice that this limit is singular even without the presence of boundaries: the Navier-Stokes equations are semilinear and, at leading order in derivatives, has a parabolic-elliptic character whereas the Euler equations arer quasilinear and have a hyperbolic-elliptic character.

The limit $\mathbf{Re} \to 0$ is the *low Reynolds number limit* and the formal limit is the (quasi-static) Stokes equations

$$\begin{aligned} 0 &= -\nabla p + \Delta u \\ \nabla \cdot u &= 0. \end{aligned}$$

Note that this limit is less singular than the high Reynolds number limit. It is the former, the high Reynolds number limit, that we will be most concerned with in this course.

2.2 Scaling symmetry

First, for any $\lambda > 0$, if u, p solve the Navier-Stokes equations, then so does

$$\begin{aligned} u_\lambda(t, x) &= \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \\ p_\lambda(t, x) &= \frac{1}{\lambda^2} p\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right). \end{aligned}$$

As above, if we are considering the problem on a domain with boundaries, then these must be scaled with respect to λ as well. This is called a *scaling invariance* or a *scaling symmetry*. One very important thing to note is that in 2D:

$$\|u_\lambda(t)\|_{L^2} = \left\|u(t\lambda^{-2})\right\|_{L^2},$$

which means that the scaling symmetry preserves the kinetic energy, whereas in 3D

$$\|u_\lambda(t)\|_{L^2} = \lambda^{1/2} \left\|u(t\lambda^{-2})\right\|_{L^2},$$

and hence the kinetic energy is not preserved by the kinetic energy (instead, it is the L^3 norm which is preserved). This shows that in 3D, as $\lambda \to 0$ the length scale would go to zero and the kinetic energy goes to zero so that smaller fluid motions require less kinetic energy to sustain. This suggests that knowing the kinetic energy of the fluid is finite does not control the characteristic length-scale of fluid motions from below, in particular, boundedness of the kinetic energy is unlikely to give us much control on the derivatives. This property is called *energy supercritical* and it is a property of the 3D the Navier-Stokes equations (whereas in 2D it is *energy critical*). This has important implications.

2.3 Boundary conditions

Unlike the Euler equations, the Navier-Stokes equations are *semi-linear* and at the highest order in derivatives in space and time, parabolic-elliptic (though the equations also have a hyperbolic character to them). As we know from basic theory [Eva98], parabolic equations generally require different boundary conditions than hyperbolic equations, and the same is true here. In the Euler equations, recall that forces were only normal to fluid elements and it made sense to use no-penetration conditions along boundaries: $u \cdot n|_{\partial\Omega} = 0$. This allows, for example, layers of fluid to slip along the boundaries of domains or immersed objects without giving rise to any forces. However, in the Navier-Stokes equations there is friction when two layers of fluid slide past one another, and so no-penetration conditions no longer make as much sense. Instead, we usually have *no-slip* conditions along solid boundaries:

$$u|_{\partial\Omega} = 0,$$

that is, the fluid must be at rest at the boundary with a stationary object immersed in the fluid (stationary in the inertial frame, so the condition could be non-zero Dirichlet conditions in a different frame). This means that along the wing of an airplane, for example, if you go close enough to the wing, the fluid is basically stuck to the wing. A few millimeters away from the edge, however, the fluid is going a few hundred miles an hour in the frame of the airplane. Hence, there is a massive gradient in velocity field in the layer along the wings and we can guess that, despite the insanely high Reynolds number, viscosity cannot be neglected near this layer along the wing and indeed, we will see that without this layer, the plane cannot fly. The precise formulation of this below is from [BT13].

Theorem 2.1 (d'Alembert's paradox)**.** *Let $\Omega \subset \mathbb{R}^3$ be a compact object with a smooth boundary which is topologically isomorphic to the closed ball immersed in an infinite ideal, incompressible fluid. Suppose that the ideal fluid is stationary, irrotational, and constant at infinity. That is, $u(t,x) = u(x)$, $x \in \mathbb{R}^3 \setminus \Omega$, $\lim_{|x|\to\infty} u(x) = u_\infty$ supplemented with no-penetration conditions $u \cdot n|_{\partial\Omega} = 0$ and that $u \in C^\infty(\mathbb{R}^3 \setminus \Omega)$. Prove that the net force on Ω is zero. In particular, there cannot be any net drag or lift on the object and so flight is impossible. This is known as* d'Alembert's paradox.

Proof of Theorem 2.1. Since $\nabla \times u = 0$ everywhere in the fluid, we have that Bernoulli's law holds:

$$\nabla \left(\frac{|u|^2}{2} + p \right) = 0.$$

As $\mathbb{R}^3 \setminus \Omega$ is simply connected (recall "simply connected" means every closed curve can be continuously contracted to a point), the fluid velocity is the gradient of a harmonic function, known as potential flow:

$$\begin{aligned} u &= -\nabla\phi \\ -\Delta\phi &= 0. \end{aligned}$$

Note that the no-penetration conditions imply the homogeneous Neumann conditions $\nabla\phi \cdot n|_{\partial\Omega} = 0$ and the boundary conditions at infinity imply

$$\lim_{|x|\to\infty} \phi(x) = u_\infty \cdot x + C,$$

for some arbitrary constant C which we can set to zero.

The total force exerted on the object is:

$$F = \int_{\partial\Omega} pn(x) dS(x),$$

where n is the unit outward normal. Let $R > 0$ be chosen large enough such that $\Omega \subset B_R(0)$. By the no penetration conditions and the fact that (u, p) solve the stationary Euler equations:

$$\begin{aligned} F &= -\int_{\partial\Omega} ((u\cdot n)u + pn)\, dS \\ &= -\int_{\partial\Omega} ((u\cdot n)u + pn)\, dS + \int_{|x|=R} \left((u\cdot \frac{x}{R})u + p\frac{x}{R}\right) dS \\ &\quad - \int_{|x|=R} \left((u\cdot \frac{x}{R})u + p\frac{x}{R}\right) dS \\ &= -\int_{B_R\setminus\Omega} \nabla\cdot(u\otimes u) + \nabla p dx - \int_{|x|=R} \left((u\cdot \frac{x}{R})u + p\frac{x}{R}\right) dS \\ &= -\int_{|x|=R} \left((u\cdot \frac{x}{R})u + p\frac{x}{R}\right) dS. \end{aligned} \tag{2.2}$$

It remains to demonstrate that the remaining term vanishes as $R \to \infty$.

Exercise 2.2. Prove that (for some constant p_∞ which you can determine).

$$\lim_{|x|\to\infty} |u(x) - u_\infty| \lesssim \frac{1}{|x|^3}$$
$$\lim_{|x|\to\infty} |p(x) - p_\infty| \lesssim \frac{1}{|x|^3}.$$

Use this to prove that the remaining term in (2.2) goes to zero as $R \to \infty$.

□

Remark 2.3. Theorem 2.1 is not true in 2D. Note that $\mathbb{R}^2 \setminus K$ is not simply connected in 2D and that the Green's function to $-\Delta$ has much slower decay at large values of x than in 3D. Hence in 2D one can generate lift without viscosity. One may be tempted to use a 2D approximation for flow over a wing but it is important to keep in mind that 2D laminar flow configurations are generally unstable to 3D perturbations at sufficiently high Reynolds number (see e.g. [Dra02, Yag12] or week two of these notes).

The small layer of fluid around an immersed object or boundary where viscosity is important is called a *boundary layer*. Boundary layers are very crucial for understanding high Reynolds number limits in the presence of boundaries and immersed ojbects, for example, at all high Reynolds number, vorticity can be generated at the boundaries whereas at infinite Reynolds number, it cannot. No-slip conditions are definitely not the only possible boundary conditions. One example is at the surface between two (approximately) immiscible fluids, such as water and air. At this boundary there is a stress balance and the possible involvement of surface tension. Still other examples exist, for example those which permit a certain amount of slip, but the no-slip conditions are by far the most dominant in application, despite being relatively problematic at high Reynolds number, and it is on these we will focus.

2.4 Vorticity, conserved, dissipated quantities

As in §1.14, we are again only considering smooth solutions of the Navier-Stokes equations (2.1a). The first thing to note is that the vorticity equations still make sense for the Navier-Stokes equations:

$$\partial_t\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u + \frac{1}{\mathbf{Re}}\Delta\omega. \tag{2.3}$$

In the absence of boundaries, e.g. for $\mathbb{R}^d$ and $\mathbb{T}^d$, we still have the conservation of mean velocity and vorticity:

$$\int u(t,x)dx = \int u(0,x)dx$$
$$\int \omega(t,x)dx = \int \omega(0,x)dx.$$

However, the kinetic energy $E(t) = \frac{1}{2}\int |u(t,x)|^2\, dx$ is not conserved but rather dissipated. Indeed, using the no-slip boundary conditions we have

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_\Omega u_j u_j dx &= -\int_\Omega u_j\partial_j p dx - \int_\Omega u_j u_i \partial_i u_j dx + \frac{1}{\mathbf{Re}}\int u_j \partial_{ii} u_j dx \\
&= -\int_\Omega u\cdot\nabla p dx - \frac{1}{2}\int_\Omega u\cdot\nabla(|u|^2)dx - \frac{1}{\mathbf{Re}}\int_\Omega \partial_i u_j \partial_i u_j dx + \frac{1}{\mathbf{Re}}\int_{\partial\Omega} u_j\partial_i u_j n_i dS \\
&= -\frac{1}{\mathbf{Re}}\int_\Omega |\nabla u|^2\, dx. \qquad (2.4)
\end{aligned}$$

This implies the energy balance (for smooth solutions),

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{\mathbf{Re}}\int_0^t \|\nabla u(\tau)\|_{L^2}^2\, d\tau = \frac{1}{2}\|u(0)\|_{L^2}^2\,. \qquad (2.5)$$

Exercise 2.4. Consider smooth solutions to 2D Navier-Stokes on $\mathbb{T}^2$ and assume for now that the solution is global and remains smooth (we will prove this later). Prove that if $\int_{\mathbb{T}^2} u(0,x)dx = 0$, then there a constant $c_0 > 0$ such that

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2}\, e^{-c_0\frac{t}{\mathbf{Re}}}.$$

Note that this estimate only predicts appreciable decay after $t \gtrsim \mathbf{Re}$. The limits $t\to\infty$ and $\mathbf{Re}\to\infty$ are hence trivially non-commutative for the initial value problem: ideal fluids do not decay whereas viscous fluids do.

In the absence of boundaries in 2D, we have a similar balance for the enstrophy using the vorticity transport equations:

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \frac{1}{\mathbf{Re}}\|\nabla\omega(t)\|_{L^2}^2 = 0.$$

More generally,

$$\frac{1}{p}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \frac{4(p-1)}{p^2\mathbf{Re}}\left\|\nabla\omega(t)^{p/2}\right\|_{L^2}^2 = 0$$

However, at the boundaries, vorticity can be generated in a fluid at finite Reynolds number whereas it cannot at infinite Reynolds number.

Exercise 2.5. Using the Gagliardo-Nirenberg-Sobolev-type inequality

$$\|f\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^1(\mathbb{R}^2)}^{1/2}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/2}$$

to prove that all smooth, global solutions to the 2D Navier-Stokes equations in $\mathbb{R}^2$ with $\omega(0)\in L^1\cap L^2(\mathbb{R}^2)$ satisfy the following for some $C = C(\|\omega(0)\|_{L^1\cap L^2})$,

$$\|\omega(t)\|_{L^2} \leq \min\left(C\left(\frac{\mathbf{Re}}{t}\right)^{1/2}, \|\omega(0)\|_{L^2}\right).$$

Other conservation laws for the Euler equations do not always fare as well as the energy and enstrophy, for example, the helicity satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^3} u(t,x)\cdot\omega(t,x)dx = -\frac{2}{\mathbf{Re}}\int \partial_j u_i \partial_j \omega_i dx,$$

which in general is not sign definite.

Exercise 2.6. See what happens to the other conserved quantities for Euler when viscosity is added (say in $\mathbb{R}^d$ and $\mathbb{T}^d$).

Recall that in the 3D Euler equations, the topology of vortex tubes and lines does not change in time, from Theorem 1.33. In the 3D Navier-Stokes equations this is *not* true: vortex tubes can and do change topology as time goes forward. At high Reynolds number, this tends to happen when large amounts of vorticity are pointing in different directions in a small region (for example, two vortex filaments getting close to one another). The rearrangement of the topology can sometimes be sudden and a bit violent – it is known as *vortex reconnection* (for reasons which will be more clear after a few youtube videos are watched). After it occurs, the solutions to Navier-Stokes and Euler will be very different.

2.5 Some special solutions

As in Euler, some of the simplest solutions to the Navier-Stokes equations are shear flows. For example, in 2D, if we look for a solution of the type:

$$u(t,x,y) = \begin{pmatrix} U(t,y) \\ 0 \end{pmatrix}$$

we find that

$$\partial_t U = \frac{1}{\mathbf{Re}}\Delta U,$$

that is, U solves the 1D heat equation. Hence, unlike the Euler equations, these are normally not stationary solutions. One situation where we can get a non-trivial stationary solution is if we take a bounded channel, say $x \in \mathbb{R}$, $y \in [0,1]$, and move the top boundary at speed β. Due to the no-slip condition we would then get the boundary conditions $U(t,0) = 0$ and $U(t,1) = \beta$. Then we see that $U(t,y) = \beta y$ is a stationary soolution to the heat equation statisfying the boundary conditions. Moreover, all shear flows will relax to this flow. This shear flow is called the *Couette flow*. If one removes the boundaries and take $y \in \mathbb{R}$, we still have that $U(t,y) = \beta y$ is a solution for all β.

Similarly, in 2D, all solutions with radially symmetric vorticity solve the heat equation as well: If $\omega(t,x)$ solves the 2D Navier-Stokes equations such that

$$\omega(t,x) = f(t,|x|)$$

for some f, then

$$\partial_t \omega = \frac{1}{\mathbf{Re}}\Delta\omega.$$

An especially important example is the Oseen vortex:

$$\omega(t,x) = \frac{\mathbf{Re}}{4\pi t}e^{-\frac{\mathbf{Re}|x|^2}{4t}}.$$

One can show that all finite enstrophy solutions to the 2D Navier-Stokes equations eventually look like the Oseen vortex [GW05]. It is also relevant for understanding point-like vortices and singular initial data [GG05].

3 Local existence and uniqueness of strong solutions for Euler

The goal of this section is to establish the local existence and uniqueness of strong solutions to the Euler equations. For simplicity of the presentation, throughout this section we restrict our attention to the case of no boundaries, i.e. $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$. Unless this aspect is relevant, we shall ignore the dependance of the Lebesgue and Sobolev spaces on the spacial domains, i.e. we simply write L^p instead of $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{T}^d)$. The case of a general smooth bounded domain Ω requires a few extra ingredients, see [Tem75, KL84].

Recall cf. (1.56) that the Euler system may be written as

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) = 0, \qquad u|_{t=0} = u_0 \in L^2_\sigma. \tag{3.1}$$

The pressure gradient may be recovered from (1.43). Here $\mathbb{P}$ is the Leray projector from L^2 onto L^2_σ (space defined in (1.50)–(1.51)). In particular $\mathbb{P}$ is self-adjoint on L^2. The following result may be traced in one form or another to [Lic25, EM69, Kat72, KP88]. We closely follow the presentation in [MB02].

Theorem 3.1 (Local existence of strong solutions for Euler)**.** *Assume that $u_0 \in H^s_\sigma := L^2_\sigma \cap H^s$, where $s > d/2 + 1$. Then there exists $T_0 = T_0(\|u_0\|_{H^s}) > 0$, and a unique solution $u \in C([0, T_0); H^s_\sigma) \cap C^1([0, T_0); H^{s-1}_\sigma)$ of the Cauchy problem for the homogenous incompressible Euler equations (3.1), with initial value u_0.*

The proof of Theorem 3.1 consists of several steps:

(i) Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a standard family of mollifiers (see Definition D.1 below). For any $\varepsilon > 0$, we show that there exists $T_\varepsilon > 0$ and a unique solution

$$u^\varepsilon \in C([0, T_\varepsilon); H^s_\sigma) \cap C^1([0, T_\varepsilon); H^{s-1}_\sigma)$$

of the mollified Euler equations

$$\partial_t u^\varepsilon + \phi_\varepsilon * \mathbb{P}((\phi_\varepsilon * u^\varepsilon) \cdot \nabla(\phi_\varepsilon * u^\varepsilon)) = 0 \tag{3.2}$$

with initial condition

$$u^\varepsilon|_{t=0} = u^\varepsilon_0 = \phi_\varepsilon * u_0 \in L^2_\sigma. \tag{3.3}$$

(Note that this type of approximation procedure is not suitable for a bounded domain.)

(ii) For each $\varepsilon > 0$ we may take the maximal time of existence of u^ε in H^s as $T_\varepsilon = \infty$.

(iii) There exists a $T_0 > 0$ such that the family u^ε obeys uniform in ε bounds in

$$L^\infty([0, T_0]; H^s_\sigma) \cap \mathrm{Lip}([0, T_0]; H^{s-1}_\sigma).$$

(iv) The family u^ε is Cauchy in the topology of $C([0, T_0]; L^2_\sigma)$, and thus has a limit point u in this space. Additionally u inherits regularity properties from the family u^ε.

(v) The limiting object u solves the initial value problem (3.1), and is the unique solution in this class.

The rest of this section consists in proving the above four points, thus yielding the proof of Theorem 3.1.

3.1 Proof of (i): local existence of solutions for mollified Euler

We write the mollification operator $\mathcal{J}_\varepsilon f = \phi_\varepsilon * f$, and denote the nonlinearity in (3.2) as

$$F_\varepsilon(u) = -\mathcal{J}_\varepsilon \mathbb{P}(\mathcal{J}_\varepsilon u \cdot \nabla \mathcal{J}_\varepsilon u).$$

The mollified Euler equation then may be written as an ODE with respect to time with value in the Banach space H^s_σ:

$$\frac{du^\varepsilon}{dt} = F_\varepsilon(u^\varepsilon), \qquad u_0^\varepsilon = \mathcal{J}_\varepsilon u_0 = \mathbb{P}\mathcal{J}_\varepsilon u_0. \tag{3.4}$$

where we use that the mollification operator $\mathcal{J}_\varepsilon = \phi_\varepsilon *$ commutes with $\mathbb{P}$ (as they are both Fourier multipliers), and that the initial datum belongs to H^s_σ. In order to show that (3.4) has a solution, we show that its integrated version has a solution, and consider the operator Φ_ε defined on $C((-T_\varepsilon, T_\varepsilon); H^s_\sigma)$ by

$$\Phi_\varepsilon(u)(t) = u_0^\varepsilon + \int_0^t F_\varepsilon(u(\tau))d\tau$$

where $T_\varepsilon > 0$ is to be determined later on. Our goal will be to show that Φ_ε is a bounded map on this space, and moreover that it is a contraction. The Banach fixed point theorem will when yield a unique fixed point u^ε for the map $u \mapsto \Phi_\varepsilon(u)$.

For this purpose we first show that

$$F_\varepsilon\left(B_{H^s_\sigma}(R)\right) \to B_{H^s_\sigma}\left(\frac{C_0 R^2}{\varepsilon}\right) \tag{3.5}$$

for some positive constant $C_0 > 0$ that is independent of R and ε. where $B_H(R)$ denotes the ball of radius R in the Banach space H. In order to prove (3.5) we use that $s > d/2$, which implies that H^s_σ is an algebra (that is, $\|uv\|_{H^s} \leq C(d,s)\|u\|_{H^s}\|v\|_{H^s}$ holds for $u, v \in H^s$), and properties of the mollifier (cf. Proposition D.5), to obtain

$$\begin{aligned}
\|F_\varepsilon(u)\|_{H^s} &= \|\mathbb{P}\nabla \cdot \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon u \otimes \mathcal{J}_\varepsilon u)\|_{H^s} \\
&\leq C\varepsilon^{-1}\|\mathcal{J}_\varepsilon u \otimes \mathcal{J}_\varepsilon u\|_{H^s} \\
&\leq C\varepsilon^{-1}\|u\|_{H^s}^2
\end{aligned}$$

where the constant C is independent of ε. Next, we show that F_ε is locally Lipschitz continuous on H^s_σ, and establish that

$$\sup_{u,v \in B_{H^s_\sigma}(R)} \frac{\|F_\varepsilon(u) - F_\varepsilon(v)\|_{H^s}}{\|u - v\|_{H^s}} \leq \frac{C_0 R}{\varepsilon} \tag{3.6}$$

where $C_0 > 0$ is the same constant as before, and in particular it is independent of ε and R. To see this, notice that since $\nabla, \mathbb{P}$, and $\mathcal{J}_\varepsilon$ are linear operators, we have

$$\begin{aligned}
\|F_\varepsilon(u) - F_\varepsilon(v)\|_{H^s} &= \|\mathbb{P}\nabla \cdot \mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon u \otimes \mathcal{J}_\varepsilon u - \mathcal{J}_\varepsilon v \otimes \mathcal{J}_\varepsilon v)\|_{H^s} \\
&\leq C\varepsilon^{-1}\left(\|(\mathcal{J}_\varepsilon u - \mathcal{J}_\varepsilon v) \otimes \mathcal{J}_\varepsilon u\|_{H^s} + \|\mathcal{J}_\varepsilon v \otimes (\mathcal{J}_\varepsilon u - \mathcal{J}_\varepsilon v)\|_{H^s}\right) \\
&\leq C\varepsilon^{-1}\left(\|\mathcal{J}_\varepsilon u\|_{H^s} + \|\mathcal{J}_\varepsilon v\|_{H^s}\right)\|u - v\|_{H^s} \\
&\leq C\varepsilon^{-1}\left(\|u\|_{H^s} + \|v\|_{H^s}\right)\|u - v\|_{H^s}
\end{aligned}$$

where the constant $C > 0$ is independent of ε. Note that the Lipschitz constant in (3.6) may also be written as $C\varepsilon^{-1-s}\left(\|u\|_{L^2} + \|v\|_{L^2}\right)$, which may become useful in view of the conservation of energy.

We now may conclude as follows. Let $u_0 \in H^s_\sigma$, and define

$$R = 2\|u_0\|_{H^s}, \qquad T_\varepsilon = \frac{\varepsilon}{2C_0 R},$$

where C_0 is the same constant as in (3.5) and (3.6). Next, consider the *complete* Banach space

$$\mathcal{B} = C([-T_\varepsilon, T_\varepsilon]; \bar{B}_{H^s_\sigma(R)})$$

where $\bar{B}$ denotes the closed ball. We next claim that $\Phi_\varepsilon \colon \mathcal{B} \to \mathcal{B}$ is a contraction. The fact that the range lies in $\mathcal{B}$ follows from (3.6):

$$\|\Phi_\varepsilon(u)\|_{\mathcal{B}} \le \|u_0\|_{H^s} + \sup_{t\in[-T_\varepsilon,T_\varepsilon]} \int_0^t \|F_\varepsilon(u(\tau))\|_{H^s} d\tau \le \frac{R}{2} + T_\varepsilon \frac{C_0 R^2}{\varepsilon} = R.$$

The fact that Φ_ε is a contraction on $\mathcal{B}$ follows from (3.6):

$$\|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)\|_{\mathcal{B}} \le \sup_{t\in[-T_\varepsilon,T_\varepsilon]} \int_0^t \|F_\varepsilon(u(\tau)) - F_\varepsilon(v(\tau))\|_{H^s} d\tau \le T_\varepsilon \frac{C_0 R}{\varepsilon} = \frac{1}{2}.$$

From the Contraction Mapping Principle (see Theorem I.1) that there exists a unique fixed point $u^\varepsilon \in \mathcal{B}$ of the map $u \to \Phi_\varepsilon(u)$.

Upon taking a time derivative, we have thus obtained a local in time solution of (3.4), and on this time interval the size of the solution is at most double that of its initial datum. Note moreover, that in view of (3.5) the solution u^ε moreover lies in $\mathrm{Lip}((-T_\varepsilon, T_\varepsilon); H^s_\sigma)$.

3.2 Proof of (ii): global existence of solutions for mollified Euler

Note that by definition, the time of existence obtained in the previous subsection obeys $T_\varepsilon \to 0$ as $\varepsilon \to 0$. In this subsection we show that the maximal time of existence may be taken to be independent of ε, and more precisely it is infinite. For this purpose, in view of Exercise A.6, this is equivalent to showing that

$$\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^s} \le C(\|u_0\|_{H^s}, \varepsilon, T) < \infty$$

for any $T > 0$. This fact follows from the energy conservation in the mollified Euler equations. Indeed, upon taking an L^2 inner product of (3.4) with u^ε, using that $\mathbb{P}$ is self adjoint and in fact the identity on L^2_σ, and that $\mathcal{J}_\varepsilon$ is self-adjoint on L^2, we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2 &= \int F_\varepsilon(u_\varepsilon) \cdot u_\varepsilon dx \\
&= -\int \mathbb{P}\mathcal{J}_\varepsilon u^\varepsilon \cdot (\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon)\, dx \\
&= -\frac{1}{2}\int \mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \left(|\mathcal{J}_\varepsilon u^\varepsilon|^2\right) dx = 0.
\end{aligned}$$

The above conservation and properties of the mollifier show that

$$\|u^\varepsilon(t)\|_{L^2} = \|u_0^\varepsilon\|_{L^2} \le \|u_0\|_{L^2}, \qquad \text{for all } \varepsilon > 0. \tag{3.7}$$

Also, the reason why we applied an exterior mollification to the nonlinearity in Euler now becomes apparent.

Using this fact, upon integrating (3.4) in time, using properties of mollifiers, and the Sobolev product rule (see Propositin G.8)

$$\|uv\|_{H^s} \le C(d,s)\left(\|u\|_{H^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^s}\right),$$

we arrive at

$$\begin{aligned}
\|u^\varepsilon(t)\|_{H^s} &\le \|u_0\|_{H^s} + \int_0^t \|F_\varepsilon(u^\varepsilon(\tau))\|_{H^s} d\tau \\
&\le \|u_0\|_{H^s} + C\varepsilon^{-1}\int_0^t \|\mathcal{J}_\varepsilon u^\varepsilon(\tau)\|_{L^\infty}\|u^\varepsilon(\tau)\|_{H^s} d\tau \\
&\le \|u_0\|_{H^s} + C\varepsilon^{-1-d/2}\int_0^t \|\mathcal{J}_\varepsilon u^\varepsilon(\tau)\|_{L^2}\|u^\varepsilon(\tau)\|_{H^s} d\tau \\
&\le \|u_0\|_{H^s} + C\varepsilon^{-1-d/2}\|u_0\|_{L^2}\int_0^t \|u^\varepsilon(\tau)\|_{H^s} d\tau
\end{aligned}$$

for some constant $C > 0$ that is independent of t and ε. The integral form of the Grönwall inequality now implies

$$\|u^\varepsilon(t)\|_{H^s} \le \|u_0\|_{H^s}\exp\left(Ct\varepsilon^{-1-d/2}\|u_0\|_{L^2}\right) < \infty$$

for any $t > 0$, concluding the proof of global existence.

3.3 Proof of (iii): uniform bounds of the family of approximate solutions

While in the previous subsection we have obtained the global existence of solutions to (3.4), except for the L^2 estimate (3.7), which on itself is not sufficiently strong, all the bounds we have obtained depend in a bad way on ε, as $\varepsilon \to 0$. In order to establish the compactness of the family $\{u^\varepsilon\}_{\varepsilon>0}$ and pass to a converging subsequence, we need uniform in ε bounds in $\dot{H}^s$.

When $s = m$ is an integer, we may apply the Leibniz rule and obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|_{\dot{H}^m}^2 &= \sum_{|\alpha|=m}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u^\varepsilon\|_{L^2}^2 \\
&= -\sum_{|\alpha|=m}\int \mathbb{P}\partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon \cdot \partial^\alpha(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon)dx \\
&= -\sum_{|\alpha|=m}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\int \partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon \cdot (\partial^{\alpha-\beta}\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla\partial^\beta \mathcal{J}_\varepsilon u^\varepsilon)dx
\end{aligned}$$

upon again using that $\mathbb{P}$ is self adjoint and that $\mathbb{P}u^\varepsilon = u^\varepsilon$. The key observation to make here is that there is no contribution from the case $\beta = \alpha$:

$$\int \partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon \cdot (\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla\partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon)dx = \frac{1}{2}\int \mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla\left|\partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon\right|^2 dx = 0$$

since $\nabla\cdot\mathcal{J}_\varepsilon u^\varepsilon = \mathcal{J}_\varepsilon\nabla\cdot u^\varepsilon = 0$. From the Hölder inequality and the Gagliardo-Nirenberg inequality (which follows from the argument used to prove Proposition G.5 and optimization in a frequency cutoff)

$$\|f\|_{\dot{W}^{i,\frac{2m}{i}}} \le C\|f\|_{L^\infty}^{1-\frac{i}{m}}\|f\|_{\dot{H}^m}^{\frac{i}{m}} \tag{3.8}$$

which holds for any $f \in L^\infty \cap H^m$ and any integer $0 \leq i \leq m$, we thus obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^\varepsilon(t)\|_{H^m}^2 &\leq \sum_{|\alpha|=m}\sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\int |\partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon|\,|\partial^{\alpha-\beta}\mathcal{J}_\varepsilon u^\varepsilon|\,|\nabla\partial^\beta \mathcal{J}_\varepsilon u^\varepsilon|dx \\
&\leq C\sum_{|\alpha|=m}\sum_{0\leq\beta<\alpha}\|\partial^\alpha \mathcal{J}_\varepsilon u^\varepsilon\|_{L^2}\|\partial^{\alpha-\beta}\mathcal{J}_\varepsilon u^\varepsilon\|_{L^{\frac{2(m-1)}{m-1-|\beta|}}}\|\nabla\partial^\beta \mathcal{J}_\varepsilon u^\varepsilon\|_{L^{\frac{2(m-1)}{|\beta|}}} \\
&\leq C\|\mathcal{J}_\varepsilon u^\varepsilon\|_{\dot{H}^m}\sum_{k=0}^{m-1}\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{\dot{W}^{m-1-k,\frac{2(m-1)}{m-1-k}}}\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{\dot{W}^{k,\frac{2(m-1)}{k}}} \\
&\leq C\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|\mathcal{J}_\varepsilon u^\varepsilon\|_{\dot{H}^m}^2
\end{aligned} \tag{3.9}$$

for some constant $C > 0$ that may depend on dimension and m, but not on ε.

Remark 3.2. Even if s is not an integer, the estimate (3.9) still holds. For this purpose, as before we write

$$\langle u^\varepsilon, F_\varepsilon(u^\varepsilon)\rangle_{\dot{H}^s} = \langle \mathcal{J}_\varepsilon u^\varepsilon, \mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon\rangle_{\dot{H}^s}$$

and then appeal to Fourier analysis to show that if $u \in H^s_\sigma$, since $\nabla \cdot u = 0$ we have

$$|\langle u, u\cdot\nabla u\rangle_{\dot{H}^s}| \leq C\|u\|_{H^s}^2\|\nabla u\|_{L^\infty} \tag{3.10}$$

for some $C = C(d, s) > 0$. Note that from the mean value theorem one immediately obtains the slightly weaker bound

$$\begin{aligned}
|\langle u, u\cdot\nabla u\rangle_{\dot{H}^s}| &= (2\pi)^d\left|\iint |\xi|^{2s}\widehat{u}_j(\xi)\widehat{u}_k(\xi-\eta)\eta_k\overline{\widehat{u}_j}(\eta)d\eta d\xi\right| \\
&= (2\pi)^d\left|\iint |\xi|^{s}(|\xi|^s-|\eta|^s)\widehat{u}_j(\xi)\widehat{u}_k(\xi-\eta)\eta_k\overline{\widehat{u}_j}(\eta)d\eta d\xi\right| \\
&\leq C\iint |\xi|^s|\widehat{u}_j(\xi)|(|\xi-\eta|^{s-1}+|\eta|^{s-1})|\xi-\eta|\,|\widehat{u}_k(\xi-\eta)|\,|\eta|\,|\overline{\widehat{u}_j}(\eta)|d\eta d\xi \\
&\leq C\|u\|_{\dot{H}^s}^2\|\widehat{\nabla u}\|_{L^1}
\end{aligned}$$

for some positive constant $C > 0$ that depends only on d and s. Note however that $\|\nabla u\|_{L^\infty} \leq \|\widehat{\nabla u}\|_{L^1}$, and this is why the bound is weaker. Proving (3.10) requires a slightly more delicate argument.

Exercise 3.3. Read the proof of (3.10) in the case of a real number s, from [KP88, Lemma X1, Appenix].

The upshot of (3.7) combined with (3.9), or more generally with (3.10), is that we have the bound

$$\frac{d}{dt}\|u^\varepsilon\|_{H^s}^2 \leq C\|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^s}^2 \leq C\|\nabla u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^s}^2 \tag{3.11}$$

for some $C = C(s, d) > 0$. At this point the assumption

$$s > d/2 + 1$$

essentially comes in, as it ensures via the Sobolev embedding that

$$H^s \subset W^{1,\infty}. \tag{3.12}$$

Note that the proof of the above embedding in fact also yields the embedding $\|\widehat{\nabla u}\|_{L^1} \le C\|u\|_{H^s}$, and thus the bound established in Remark 3.2 above will also be sufficient for our purposes. The upshot of the Sobolev embedding (3.12) and of the nonlinear estimate (3.11) is that

$$\frac{d}{dt}\|u^\varepsilon\|_{H^s}^2 \le C_1\|u^\varepsilon\|_{H^s}^3 \tag{3.13}$$

holds for some constant $C_1 = C_1(d,s) > 0$. Upon integrating (3.13) and using the comparison principle for ODEs we obtain thus that

$$\|u^\varepsilon\|_{L^\infty([0,T_0];H^s)} \le 2\|u_0\|_{H^s}, \qquad \text{for all } \varepsilon > 0, \tag{3.14}$$

where

$$T_0 = \frac{1}{C_1\|u_0\|_{H^s}} > 0$$

and C_1 is the constant from (3.13).

We note that a proof similar to that of (3.5) shows that

$$\sup_{t\in[0,T_0]}\|F_\varepsilon(u^\varepsilon(t))\|_{H^{s-1}} \le C \sup_{t\in[0,T_0]}\|u^\varepsilon(t)\|_{H^s}^2 \le 4C\|u_0\|_{H^s}^2$$

and thus

$$\|u^\varepsilon\|_{\mathrm{Lip}([0,T_0];H^{s-1})} \le 4C\|u_0\|_{H^s}^2, \qquad \text{for all } \varepsilon > 0 \tag{3.15}$$

where C is independent of ε.

3.4 Proof of (iv): compactness of the family of approximate solutions

From Banach-Alaoglu, it follows from (3.14) that the family $\{u^\varepsilon\}$ contains at least one *weak-* convergent* subsequence. Moreover, using that whenever $x_n \rightharpoonup x$ we have $\|x\| \le \liminf_{n\to\infty}\|x_n\|$, any weak-* limit point of the family $\{u^\varepsilon\}$ also obeys the bound (3.14). We are however after strong convergence.

Since by the Rellich compactness theorem the embedding $H^s \subset H^{s-1}$ is compact on $\mathbb{T}^d$, and locally compact on $\mathbb{R}^d$, the uniform in ε bounds (3.14) and (3.15) are sufficient to apply a generalization of the *Arzela-Ascoli theorem* (called Aubin-Lions compactness therorem, see Theorem C.6 below), and conclude that the sequence u^ε is pre-compact in $C([0,T_0];H^{s-1}(\mathbb{T}^d))$, respectively in $C([0,T_0];H^{s-1}(\mathbb{R}^d_{\mathrm{loc}}))$.

We however choose to pursue an alternative and slightly less technical route in order to obtain a strongly convergent subsequence of the family $\{u^\varepsilon\}_{\varepsilon>0}$. (Re-vist this once more). We claim that for $0<\varepsilon,\delta\in(0,1]$ we have

$$\|u^\varepsilon - u^\delta\|_{L^\infty(0,T_0;L^2)} \le C\min\{\varepsilon,\delta\}\|u_0\|_{H^s} \tag{3.16}$$

and thus that

$$\{u^\varepsilon\} \text{ is Cauchy in } C([0,T_0];L^2) \tag{3.17}$$

where the continuity in time follows from (3.15) (since $s\ge 1$). Note that $C([0,T_0];L^2)$ is complete, so that (3.17) guarantees that there exists $\varepsilon_n \to 0$ and

$$u \in C([0,T_0];L^2)\cap \mathrm{Lip}([0,T_0];H^{s-1})\cap L^\infty([0,T_0];H^s) \tag{3.18}$$

such that

$$u^{\varepsilon_n} \to u \text{ in } C([0,T_0];L^2), \text{ as } n \to \infty, \tag{3.19}$$

$$u^{\varepsilon_n} \rightharpoonup^* u \text{ in } L^\infty([0,T_0];H^s), \text{ as } n \to \infty. \tag{3.20}$$

In fact, the Sobolev interpolation inequality

$$\|u\|_{H^r} \le \|u\|_{L^2}^{1-r/s}\|u\|_{H^s}^{r/s}$$

which holds for $0 \le r \le s$, shows that

$$u^{\varepsilon_n} \to u \text{ in } C([0,T_0];H^r), \text{ as } n \to \infty, \tag{3.21}$$

for any $0 \le r < s$. The strong convergence in (3.21) is relevant since from the fact that $s > d/2+1$, we may find an $r \in (d/2+1, s)$, and thus

$$\nabla u^{\varepsilon_n} \to \nabla u \text{ in } C([0,T_0];L^\infty), \text{ as } n \to \infty, \tag{3.22}$$

fact that will be convenient in the next section.

The argument in this section is thus concluded if we prove that (3.16) holds. The main ingredient here is a property of mollifiers: the bound

$$\|(\mathcal{J}_\varepsilon - \mathcal{J}_\delta)f\|_{H^r} \le C\min\{\varepsilon,\delta\}\|f\|_{H^{r+1}}$$

holds for all $r \ge 0$ an some positive constant $C > 0$, which is independent of ε and δ. We first write

$$\begin{aligned}
F_\varepsilon(u^\varepsilon) - F_\delta(u^\delta) &= \mathbb{P}\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) - \mathbb{P}\mathcal{J}_\delta(\mathcal{J}_\delta u^\delta \cdot \nabla \mathcal{J}_\delta u^\delta) \\
&= \mathbb{P}(\mathcal{J}_\varepsilon - \mathcal{J}_\delta)(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) + \mathbb{P}\mathcal{J}_\delta((\mathcal{J}_\varepsilon - \mathcal{J}_\delta)u^\varepsilon \cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) \\
&\quad + \mathbb{P}\mathcal{J}_\delta(\mathcal{J}_\delta(u^\varepsilon - u^\delta)\cdot \nabla \mathcal{J}_\varepsilon u^\varepsilon) + \mathbb{P}\mathcal{J}_\delta(\mathcal{J}_\delta u^\delta \cdot \nabla(\mathcal{J}_\varepsilon - \mathcal{J}_\delta)u^\varepsilon) \\
&\quad + \mathbb{P}\mathcal{J}_\delta(\mathcal{J}_\delta u^\delta \cdot \nabla \mathcal{J}_\delta(u^\varepsilon - u^\delta)).
\end{aligned}$$

We take an L^2 inner product of the above with $u^\varepsilon - u^\delta$, and note that the contribution from the last term in the above vanishes (since $\nabla \cdot \mathcal{J}_\delta u^\delta = 0$). Moreover, the remaining four terms all contain the operator $\mathcal{J}_\varepsilon - \mathcal{J}_\delta$, and thus we arrive at

$$\begin{aligned}
\frac{d}{dt}\|u^\varepsilon - u^\delta\|_{L^2}^2 &\le C\min\{\varepsilon,\delta\}\|u^\varepsilon - u^\delta\|_{L^2}\left(\|u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^2} + \|u^\varepsilon\|_{H^1}\|\nabla u^\varepsilon\|_{L^\infty} + \|u^\delta\|_{L^\infty}\|u^\varepsilon\|_{H^2}\right) \\
&\quad + C\|u^\varepsilon - u^\delta\|_{L^2}^2\|\nabla u^\varepsilon\|_{L^\infty}.
\end{aligned}$$

In view of (3.12) and since $s > d/2+1 \ge 2/2+1 = 2$, we thus obtain from the above estimate and the uniform bound (3.14) that

$$\begin{aligned}
\frac{d}{dt}\|u^\varepsilon - u^\delta\|_{L^2}^2 &\le C\|u^\varepsilon\|_{H^s}\|u^\varepsilon - u^\delta\|_{L^2}^2 + C\min\{\varepsilon,\delta\}\|u^\varepsilon - u^\delta\|_{L^2}\left(\|u^\varepsilon\|_{H^s}^2 + \|u^\delta\|_{H^s}^2\right) \\
&\le C\|u_0\|_{H^s}\|u^\varepsilon - u^\delta\|_{L^2}^2 + C\min\{\varepsilon,\delta\}\|u_0\|_{H^s}^2\|u^\varepsilon - u^\delta\|_{L^2}
\end{aligned} \tag{3.23}$$

for $t \in [0,T_0]$ and a constant $C > 0$ that is independent of ε and δ. Lastly we note that

$$\|u_0^\varepsilon - u_0^\delta\|_{L^2} = \|(\mathcal{J}_\varepsilon - \mathcal{J}_\delta)u_0\|_{L^2} \le C\min\{\varepsilon,\delta\}\|u_0\|_{H^1} \le C\min\{\varepsilon,\delta\}\|u_0\|_{H^s}$$

which combined with (3.23) and the Grönwall inequality yields

$$\|u^\varepsilon - u^\delta\|_{L^\infty(0,T_0;L^2)} \le C\min\{\varepsilon,\delta\}\|u_0\|_{H^s}\left(1 + T_0\|u_0\|_{H^s}\right)\exp\left(CT_0\|u_0\|_{H^s}\right)$$

which combined with the definition of T_0 as $T_0 = 1/(C_1\|u_0\|_{H^s})$ concludes the proof of (3.16).

3.5 Proof of (v): the limiting object is the unique solution of Euler in this class

We have now arrived a limit point u of the family $\{u^\varepsilon\}$, with integrability given by (3.18), and such that the limit is attained in the sense of (3.20)–(3.22). We now prove that any such limit point is a solution of the Cauchy problem for the Euler equations (3.1). For this purpose, we use the time integrated form of (3.2) and write

$$\begin{aligned}
&u(t) - u_0 + \int_0^t \mathbb{P}(u(\tau)\cdot\nabla u(\tau))d\tau \\
&\qquad = (u(t) - u^\varepsilon(t)) - (1-\mathcal{J}_\varepsilon)u_0 + \int_0^t \mathbb{P}((u(\tau)-u^\varepsilon(\tau))\cdot\nabla u(\tau))d\tau \\
&\qquad\quad + \int_0^t \mathbb{P}(u^\varepsilon(\tau)\cdot\nabla(u(\tau)-u^\varepsilon(\tau)))d\tau + \int_0^t \mathbb{P}(1-\mathcal{J}_\varepsilon)(u^\varepsilon(\tau)\cdot\nabla u^\varepsilon(\tau))d\tau \\
&\qquad\quad + \int_0^t \mathbb{P}((1-\mathcal{J}_\varepsilon)u^\varepsilon(\tau)\cdot\nabla u^\varepsilon)d\tau + \int_0^t \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon(\tau)\cdot\nabla(1-\mathcal{J}_\varepsilon)u^\varepsilon(\tau))d\tau
\end{aligned}$$

for all $t \in [0,T_0]$. We thus obtain

$$\begin{aligned}
&\left\| u(t) - u_0 + \int_0^t \mathbb{P}(u(\tau)\cdot\nabla u(\tau))d\tau \right\|_{L^2} \\
&\le \|u(t)-u^\varepsilon(t)\|_{L^2} + C\varepsilon\|u_0\|_{H^1} + \int_0^t \|u(\tau)-u^\varepsilon(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^\infty}d\tau \\
&\quad + \int_0^t \|u_0\|_{L^2}\|\nabla u(\tau)-\nabla u^\varepsilon(\tau)\|_{L^\infty}d\tau + C\varepsilon\int_0^t \|u^\varepsilon(\tau)\otimes u^\varepsilon(\tau)\|_{H^2}d\tau \\
&\quad + C\varepsilon\int_0^t \|u^\varepsilon(\tau)\|_{H^1}\|\nabla u^\varepsilon(\tau)\|_{L^\infty}d\tau + C\varepsilon\int_0^t \|u^\varepsilon(\tau)\|_{L^\infty}\|u^\varepsilon(\tau)\|_{H^2}d\tau \\
&\le \|u-u^\varepsilon\|_{L^\infty(0,T_0;L^2)}(1+CT_0\|u_0\|_{H^s}) + T_0\|u_0\|_{L^2}\|\nabla u-\nabla u^\varepsilon\|_{L^\infty(0,T_0;L^\infty)} \\
&\quad + C\varepsilon\left(\|u_0\|_{H^s} + T_0\|u_0\|_{H^s}^2\right) \\
&\le C\|u-u^\varepsilon\|_{L^\infty(0,T_0;L^2)} + C\|\nabla u-\nabla u^\varepsilon\|_{L^\infty(0,T_0;L^\infty)} + C\varepsilon\|u_0\|_{H^s}
\end{aligned} \tag{3.24}$$

for all $t \in [0,T_0]$, where in the third line of the above we have used that $\|u^\varepsilon(\tau)\|_{L^2} \le \|u_0\|_{L^2}$ for any $\tau\in[0,T_0]$, while in the last line we have appealed to (3.14) and the definition of T_0. By appealing to (3.19) and (3.22), upon sending $\varepsilon = \varepsilon_n \to 0$ in (3.24) we conclude that

$$u(t) + \int_0^t \mathbb{P}(u(\tau)\cdot\nabla u(\tau))d\tau = u_0 \tag{3.25}$$

holds for a.e. x and a.e. $t\in[0,T_0]$. Note that the a.e. equality obtained may be bootstrapped to a pointwise equality since the functions in (3.25) are continuous functions in (x,t): this in turn follows by appealing to the Sobolev embedding, (3.18), and the fact that $\mathbb{P}\colon L^2\cap C^\alpha \to L^2\cap C^\alpha$ for $\alpha\in(0,1)$, cf. Exercise 1.22. Thus, in fact (3.25) holds in the classical pointwise in (x,t) sense, which shows that u is a solution of the Euler equations (3.1) on $[0,T_0]$.

Next, we prove that there can be at most one solution u of (3.1) in the regularity class given in (3.18). If we would have two such solutions u and v, with the same initial datum u_0, then their difference would obey

$$\partial_t(u-v) + \mathbb{P}((u-v)\cdot\nabla u) + \mathbb{P}(v\cdot\nabla(u-v)) = 0, \qquad (u-v)|_{t=0} = 0.$$

Taking an L^2 inner product of the above with $u-v$, and using that $\mathbb{P}$ is self adjoint and that $\mathbb{P}(u-v) = u-v$, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 &= -\int (u-v)\cdot((u-v)\cdot\nabla u)\,dx - \int (u-v)\cdot(v\cdot\nabla(u-v))\,dx \\ &= -\int (u-v)\cdot((u-v)\cdot\nabla u)\,dx \\ &\leq \|u-v\|_{L^2}^2\|\nabla u\|_{L^\infty}.\end{aligned}$$

Therefore, as long as $u \in L^1(0,T_0;\mathrm{Lip})$, we have that it is the unique solution in this class. In our case this is again ensured by the fact that $s > d/2+1$, thereby concluding the proof of uniqueness.

The only part of the proof that remains is to show that the solution u we constructed earlier obeys

$$u \in C([0,T_0];H^s_\sigma) \tag{3.26}$$

so that we are allowed to say that the initial datum u_0 is attained in the topology of H^s. To achieve this we prove two things: first that $u \in C_w([0,T_0];H^s_\sigma)$, i.e. that $\langle u(t),\varphi\rangle_{L^2}$ is a continuous function of time for fixed test function $\varphi \in H^{-s}$; and second, that $\|u(t)\|_{H^s}$ is a continuous function of time. Together these two ingredients (weak continuity, and continuity of the norm) imply (3.26) from standard arguments.

For the first ingredient, weak continuity, fix a test function $\varphi \in H^{-s}$, with $\|\varphi\|_{H^{-s}} = 1$, and let $\delta \in (0,1/4]$ be arbitrary. We may then find an element $\psi \in H^{-s+1}$ (which is dense in H^{-s} and is the dual of H^{s-1}) such that

$$\|\varphi-\psi\|_{H^{-s}} = \left\|(1+|\xi|^2)^{-s/2}(\widehat{\varphi}(\xi)-\widehat{\psi}(\xi))\right\|_{L^2} \leq \delta$$

and such that $\|\phi\|_{H^{-s+1}} \leq C\delta^{-1}$. Due to (3.21), we have that $\langle u^\varepsilon(t),\psi\rangle_{L^2} \to \langle u(t),\psi\rangle_{L^2}$, as $\varepsilon \to 0$, uniformly in $t \in [0,T_0]$. Thus, we may find ε sufficiently small, such that

$$\sup_{t\in[0,T_0]} |\langle u(t)-u^\varepsilon(t),\psi\rangle_{L^2}| \leq \delta^2.$$

Moreover, since by construction $u^\varepsilon \in \mathrm{Lip}([0,T_0];H^{s-1})$, we have that there exists $\tau > 0$ such that

$$\|u^\varepsilon(t)-u^\varepsilon(s)\|_{H^{s-1}} \leq \delta$$

whenever $|t-s| \leq \tau$, where $t,s \in [0,T_0]$. The above three ingredients yields that for $|t-s| \leq \tau$ we have

$$\begin{aligned}|\langle u(t)-u(s),\varphi\rangle_{L^2}| &\leq |\langle u^\varepsilon(t)-u^\varepsilon(s),\psi\rangle_{L^2}| + |\langle u(t),\varphi-\psi\rangle_{L^2}| + |\langle u(t),\varphi-\psi\rangle_{L^2}| \\ &\quad + |\langle u(t)-u^\varepsilon(t),\psi\rangle_{L^2}| + |\langle u(s)-u^\varepsilon(s),\psi\rangle_{L^2}| \\ &\leq \delta\|\psi\|_{H^{-s}} + 2\delta\|u\|_{L^\infty(0,T_0;H^s)} + 2\delta^2\|\psi\|_{H^{-s+1}} \\ &\leq \delta(1-\delta) + 2\delta\|u\|_{L^\infty(0,T_0;H^s)} + C\delta.\end{aligned}$$

Since $\delta \in (0,1/4]$ was arbitrary, we conclude that u is weakly uniformly continuous with values in H^s.

For the second ingredient, in view of uniqueness and by shifting in time, we see that it is enough to check the continuity of $\|u(t)\|_{H^s}$ at $t=0$. Moreover, due to the time-reversibility of the equations it is enough to check the right continuity of the H^s norm as $t \to 0^+$. First, in view of the weak continuity established earlier, we have that

$$\liminf_{t\to 0^+} \|u(t)\|_{H^s} \geq \|u_0\|_{H^s}.$$

On the other hand, view of (3.13), for any $t > 0$ we have that

$$\|u(t)\|_{H^s} \leq \limsup_{\varepsilon\to 0} \|u^\varepsilon(t)\|_{H^s} \leq \limsup_{\varepsilon\to 0} \frac{\|u_0^\varepsilon\|_{H^s}}{1 - 2C_1 t\|u_0^\varepsilon\|_{H^s}} = \frac{\|u_0\|_{H^s}}{1 - 2C_1 t\|u_0\|_{H^s}}.$$

Taking the $\limsup_{t\to 0^+}$ of the above we arrive at

$$\limsup_{t\to 0^+} \|u(t)\|_{H^s} \leq \|u_0\|_{H^s}$$

which concludes the proof of strong continuity of the H^s norm at $t = 0$, and of the proof of Theorem 3.1.

4 The Beale-Kato-Majda criterion and global existence of 2D Euler

Next we discuss a well-known *blowup criterion*, or equivalently a *continuation criterion*, for smooth solutions to the Euler equations (it is also true for the Navier-Stokes equations, but there are better results in the viscous case as we will see). This result is due to Beale-Kato-Majda [BKM84], and we will sometimes simply refer to it as the *BKM blowup criterion*.

The main idea is as follows. Recall cf. (3.10) that when $\nabla \cdot u = 0$ and $u \in H^s$, with $s > d/2 + 1$ we have

$$\frac{d}{dt}\|u\|_{H^s}^2 = -\langle \mathbb{P}(u \cdot \nabla u), u\rangle_{H^s} = -\langle u \cdot \nabla u, u\rangle_{H^s} \le C_0 \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2 \tag{4.1}$$

for some fixed constant $C_0 = C_0(d, s) > 0$. Using the Grönwall inequality, estimate (4.1) yields

$$\|u(T)\|_{H^s}^2 \le \|u(t)\|_{H^s}^2 \exp\left(C_0 \int_t^T \|\nabla u(\tau)\|_{L^\infty} d\tau \right) \tag{4.2}$$

for $0 \le t < T$. Therefore, a control over ∇u in $L_t^1 L_x^\infty$ is sufficient to propagate smooth solutions to the Euler equations. The BKM theorem shows that we need only to control the $L_t^1 L_x^\infty$ norm of the vorticity ω, the anti-symmetric part of the gradient. It will follow from the vorticity equations that strong solutions to the Euler equations are globally well-posed in 2D.

Theorem 4.1 (Beale-Kato-Majda [BKM84])**.** *Let $u \in L^\infty([0, T); H_\sigma^s(\Omega))$ be a strong solution of the Euler equations, where $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, $d \in \{2, 3\}$, $s > d/2 + 1$, and $T > 0$. We have that*

$$\limsup_{t \to T^-} \|u(t)\|_{H^s} < \infty$$

if and only if

$$\int_0^T \|\omega(\tau)\|_{L^\infty} d\tau < \infty. \tag{4.3}$$

In short, if (4.3) holds there exists an $\eta > 0$ such that $u(t)$ can be uniquely continued as a strong solution on $[0, T + \eta)$.

There exist slightly stronger variants of Theorem 4.1, see e.g. [Pon85] where ω is replaced by the deformation tensor D_u, [KT00, KOT02] where the L^∞ norm in (4.3) is replaced by the weaker BMO or critical Besov space norm. Moreover, the results in this section also hold in the case of a domain with boundaries [Fer93]. Here we will content ourselves with the original version, which is proven on $\mathbb{R}^d$ or $\mathbb{T}^d$.

The proof of Theorem 4.1 is based on an inequality (see (4.11) below) which improves (4.1), by replacing the full ∇u matrix with the vorticity ω, at the cost of an additional logarithmic term of the H^s norm.

Before turning to this proof, we note that when $d = 2$, since the vorticity equation obeys a transport equation (cf. (1.67)) we have that $\|\omega(t)\|_{L^\infty}$ is non-increasing in time, and thus the smooth solutions cannot blow up in finite time.

Theorem 4.2 (Global existence in 2D [Wol33, Höl33, Jud63])**.** *Consider the case $d = 2$ in Theorem 4.1. Then the solution u may be continued for all time, i.e. $u \in L^\infty([0, T]; H_\sigma^s)$, for any $T > 0$. Moreover, we have the estimate*

$$\|u(t)\|_{H^s} \le (e + e\|u_0\|_{H^s})^{2\exp(Ct(1 + \|u_0\|_{L^2} + \|\omega_0\|_{L^\infty}))} \tag{4.4}$$

for all $t \ge 0$, where $C = C(s, \Omega) > 0$ is a fixed constant.

Remark 4.3. On $\mathbb{T}^2$, since u_0 has zero mean we have $\|u_0\|_{L^2} \leq C\|u_0\|_{\dot{H}^1} \leq C\|\omega_0\|_{L^2} \leq C\|\omega_0\|_{L^\infty}$, and thus the estimate (4.4) simplifies to

$$\|u(t)\|_{H^s} \leq (e + e\|u_0\|_{H^s})^{2\exp(Ct(1+\|\omega_0\|_{L^\infty}))} \tag{4.5}$$

for all $t > 0$.

Remark 4.4. On the periodic domain $\mathbb{R}^2$ or the whole space $\mathbb{T}^2$, it is not known whether the double-exponential bounds (4.4) respectively (4.5) are sharp, and only a single exponential lower bound is known for the periodic box [Zla15]. On the other hand, this bound has been established recently in the case of the bounded domain [KS13].

4.1 Boundedness of singular integral operators on L^∞ up to logarithmic errors

We consider a standard convolution type Calderón-Zygmund operator

$$Tf(x) = p.v. \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)dy \tag{4.6}$$

where the function $\Omega \colon \mathbb{S}^{d-1} \to \mathbb{R}$ is smooth and obeys the cancellation property

$$\int_{\mathbb{S}^{d-1}} \Omega(y)dy = 0.$$

Examples of such operators T are given by the singular part of the operators $\mathbb{P}$ and $\nabla u \mapsto \omega$.

It is well-known (see [Ste70]) that if $f \in L^\infty$, then we do not necessarily have $Tf \in L^\infty$. In this section we prove that the error is merely a logarithm with respect to a stronger norm f. More precisely, we establish

Theorem 4.5. *Assume that $f \in L^2 \cap L^\infty \cap C^\alpha$ for some $\alpha \in (0,1)$. Then we have that*

$$\|Tf\|_{L^\infty} \leq C\left(\|f\|_{L^2} + \|f\|_{L^\infty}\left(1 + \log_+ \frac{[f]_{C^\alpha}}{\|f\|_{L^\infty}}\right)\right) \tag{4.7}$$

holds for some positive constant $C = C(d, \alpha, \Omega)$.

Proof of Theorem 4.5. Let $\phi \colon [0,\infty) \to [0,\infty)$ be a smooth cutoff function adapted to the interval $[1,2]$, i.e. ϕ is identically 1 on $|x| \leq 1$, vanishes identically on $|x| \geq 2$, and is non-increasing. Let $R \in (0,1]$ be a smooth cutoff scale, to be determined precisely later. We then have the decomposition of $Tf(x)$ into an inner, medium, and outer piece as

$$\begin{aligned}
Tf(x) &= p.v. \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d}\phi\left(\frac{|y|}{R}\right) f(x-y)dy + \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d}\left(1-\phi\left(\frac{|y|}{R}\right)\right)\phi\left(|y|\right) f(x-y)dy \\
&\quad + \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d}\left(1-\phi\left(\frac{|y|}{R}\right)\right)\left(1-\phi\left(|y|\right)\right) f(x-y)dy \\
&= Tf_{\text{in}}(x) + Tf_{\text{med}}(x) + Tf_{\text{out}}(x).
\end{aligned}$$

Note that for any $\varepsilon > 0$ we have

$$\int_{\varepsilon \leq |y|} \frac{\Omega(y/|y|)}{|y|^d}\phi\left(\frac{|y|}{R}\right) dy = \int_\varepsilon^{2R} \frac{\phi(\rho/R)}{\rho}\int_{\mathbb{S}^{d-1}} \Omega(z)dzd\rho = 0$$

since Ω has zero mean on $\mathbb{S}^{d-1}$. Therefore, we may write

$$Tf_{\text{in}}(x) = p.v. \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} \phi\left(\frac{|y|}{R}\right) (f(x-y) - f(x))\, dy$$

and thus bound

$$\begin{aligned} |Tf_{\text{in}}(x)| &\leq \|\Omega\|_{L^\infty} [f]_{C^\alpha} \int_{|y|\leq 2R} \frac{1}{|y|^{d-\alpha}} dy \\ &\leq C[f]_{C^\alpha} R^\alpha \end{aligned} \tag{4.8}$$

where $C > 0$ is a constant, that depends as before on Ω, d, and α. On the other hand, for the medium piece we have

$$\begin{aligned} |Tf_{\text{med}}(x)| &\leq \|\Omega\|_{L^\infty} \|f\|_{L^\infty} \int_{R\leq|y|\leq 2} \frac{1}{|y|^d} dy \\ &\leq C\|f\|_{L^\infty} \log\left(\frac{2}{R}\right). \end{aligned} \tag{4.9}$$

Lastly, for the outer piece we use the decay of f and obtain that

$$\begin{aligned} |Tf_{\text{out}}(x)| &\leq \|\Omega\|_{L^\infty} \|f\|_{L^2} \left(\int_{|y|\geq 1} \frac{1}{|y|^{2d}} dy\right)^{1/2} \\ &\leq C\|f\|_{L^2}. \end{aligned} \tag{4.10}$$

Inserting

$$R^\alpha = \min\left\{1, \frac{\|f\|_{L^\infty}}{[f]_{C^\alpha}}\right\}$$

into (4.8) and (4.9) the proof of (4.7) follows. □

A number of variants of inequality (4.7) are suitable. Of importance to us will be the following:

Corollary 4.6. *Assume that $f \in H^s$ for some $s > d/2$. Then we have that*

$$\|Tf\|_{L^\infty} \leq C\left(\|f\|_{L^2} + \|f\|_{L^\infty}\left(1 + \log_+ \frac{\|f\|_{H^s}}{\|f\|_{L^\infty}}\right)\right) \tag{4.11}$$

holds for some positive constant $C = C(d, s, \Omega)$. Moreover, if $f = \nabla g$, and $g \in L^2$, we have the bound

$$\|Tf\|_{L^\infty} \leq C\left(\|g\|_{L^2} + \|f\|_{L^\infty}\left(1 + \log_+ \frac{\|f\|_{H^s}}{\|f\|_{L^\infty}}\right)\right) \tag{4.12}$$

with a possibly larger constant $C > 0$.

Proof of Corollary 4.6. In view of the Sobolev embedding $H^s \subset C^\alpha$, for any $\alpha \in (0, \min\{s - d/2, 1\})$, we have that (4.11) follows directly from (4.7). Proving (4.12) amounts to integrating by parts with respect to y in the Tf_{out} term, and noting that $\nabla_y \Omega(y/|y|) \leq C/|y|$ for all $y \neq 0$. □

4.2 Proof of the BKM criterion

In this section we give the proof of Theorem 4.1. One direction in the claimed equivalence follows from the Sobolev embedding theorem:

$$\infty = \lim_{t\to T^-} \int_0^t \|\omega(\tau)\|_{L^\infty} d\tau \le C \lim_{t\to T^-} \int_0^t \|u(\tau)\|_{H^s} d\tau$$

since $s > d/2 + 1$. The divergence of the integral directly implies that $\limsup_{t\to T^-} \|u(t)\|_{H^s} = \infty$.

For the converse, we need to show that if (4.3) holds, then we have

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \le C(T) = C\left(\|u_0\|_{H^s}, T, \int_0^T \|\omega(\tau)\|_{L^\infty} d\tau\right) < \infty, \tag{4.13}$$

for any $T > 0$. In order to achieve this we note that by (1.64) in 3D and (1.70) in 2D, we may write

$$|\nabla u(x)| \le C|\omega(x)| + |(T\omega)(x)|$$

where T is given by a Calderon-Zygmund operator of convolution type (4.6). Further, since $\omega = \nabla \times u$ in 3D, respectively $\omega = \nabla^\perp \cdot u$ in 2D, by appealing to (4.12) we have the bound

$$\|T\omega\|_{L^\infty} \le C\left(\|u\|_{L^2} + \|\omega\|_{L^\infty}\left(1 + \log_+ \frac{\|\omega\|_{H^{s-1}}}{\|\omega\|_{L^\infty}}\right)\right)$$

for some constant $C = C(d,s) > 0$. Here we used that $s - 1 > d/2$. Combining the above two estimates with (4.1), and recalling that the kinetic energy is conserved, we arrive at

$$\frac{d}{dt}\|u\|_{H^s} \le C\left(\|u_0\|_{L^2} + \|\omega\|_{L^\infty}\left(1 + \log_+ \frac{\|u\|_{H^s}}{\|\omega\|_{L^\infty}}\right)\right)\|u\|_{H^s}.$$

The above inequality implies that

$$\frac{d}{dt}\|u\|_{H^s} \le C_0\left(1 + \|u_0\|_{L^2} + \|\omega\|_{L^\infty}\left(1 + \log_+ \|u\|_{H^s}\right)\right)\|u\|_{H^s}.$$

for some constant $C_0 = C_0(d,s) > 0$. We may now compare the above estimate with the corresponding ODE which we may integrate explicitly. Indeed, for $y_0 > 0$, we have

$$\begin{aligned}
&\dot{y} = \Big(a + b(t)(1 + \log(1+y))\Big)y \\
&\Rightarrow \frac{d}{dt}(ye^{-at}) \le b(t)(1 + at + \log(1 + ye^{-at}))(ye^{-at}) \\
&\Rightarrow \frac{d}{dt}\left(\log(1 + at + \log(1 + ye^{-at}))\right) \le \frac{a}{1+at} + b(t) \\
&\Rightarrow \log(1 + at + \log(1 + y(t)e^{-at})) \le \log(1 + \log(1+y_0)) + \log(1+at) + \int_0^t b(s)ds \\
&\Rightarrow \log(1 + y(t)e^{-at}) \le (1+at)\left((1 + \log(1+y_0))\exp\left(\int_0^t b(s)ds\right) - 1\right) \\
&\Rightarrow y(t) \le e^{at}\left\{\exp\left[(1+at)\left((1 + \log(1+y_0))\exp\left(\int_0^t b(s)ds\right) - 1\right)\right] - 1\right\}
\end{aligned}$$

Setting $a = C_0(1 + \|u_0\|_{L^2})$ and $b = C_0\|\omega\|_{L^\infty}$, we arrive at

$$\|u(t)\|_{H^s} \leq \exp(A(t)) \\ \times \left\{\exp\left[(1 + A(t))\left((1 + \log(1 + \|u_0\|_{H^s}))\exp\left(C_0\int_0^t \|\omega(\tau)\|_{L^\infty} d\tau\right) - 1\right)\right] - 1\right\} \quad (4.14)$$

where

$$A(t) = C_0 t(1 + \|u_0\|_{L^2}),$$

which concludes the proof of (4.13) and thus of the theorem.

4.3 Proof of global existence in 2D

In this section we give the proof of Theorem 4.2. The key ingredient is the conservation of the L^∞ norm of the vorticity

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty} \quad (4.15)$$

which holds for all $t > 0$. In order to prove (4.15), one may either use the conservation along Lagrangian paths (1.71), or simply perform and L^p estimate on (1.67)

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p = -\int u \cdot \nabla\omega\omega|\omega|^{p-2} dx = -\frac{1}{p}\int u \cdot \nabla(|\omega|^p)\, dx = 0,$$

since $\nabla \cdot u = 0$, and pass $p \to \infty$. Combining (4.15) with the H^s inequality (4.14) established earlier, we arrive at

$$\|u(t)\|_{H^s} \leq \exp\left[2(1 + C_0 t(1 + \|u_0\|_{L^2}))(1 + \log(1 + \|u_0\|_{H^s}))\exp\left(C_0 t\|\omega_0\|_{L^\infty}\right)\right] \quad (4.16)$$

for all $t > 0$, where $C_0 = C_0(s, d) > 0$ is a constant. It is clear that (4.16) implies the bound (4.4), which concludes the proof of the theorem.