6 Leray-Hopf weak solutions

In this section we again consider the case of $\Omega = \mathbb{R}^d$, as considered by Leray [Ler34] (see also the modern discussion in [LR02]), or the slightly simpler $\Omega = \mathbb{T}^d$. The idea of the proof that we shall present in the case of the periodic domain may be modified to also deal with a bounded domain [Hop51] (see also a modern discussion in [CF88, Tem01]).

Consider a *vector valued divergence free test function*, i.e. $\phi \in C_0^\infty([0,\infty)\times\Omega)$ such that $\nabla\cdot\phi = 0$. Upon taking an L^2 inner product of (2.1a) with ϕ, and integrating by parts with respect to x and t we obtain

$$\iint_{[0,\infty)\times\Omega}\Big(u\cdot\partial_t\phi + \nu u\cdot\Delta\phi + u\cdot(u\cdot\nabla\phi)\Big)dxdt = \int_\Omega u_0\cdot\phi(\cdot,0)dx \tag{6.1}$$

where we have used that $\int_\Omega \nabla p\cdot\phi dx = -\int_\Omega p(\nabla\cdot\phi)dx = 0$. We will henceforth say that u *solves the Navier-Stokes (momentum) equation in the sense of distributions*, if (6.1) holds for any $\phi\in C_0^\infty([0,\infty)\times\Omega)$ such that $\nabla\cdot\phi = 0$.

Alternatively, we may consider $\phi\in C_0^\infty((0,\infty);\Omega)$, so that (6.1) becomes

$$\iint_{(0,\infty)\times\Omega}\Big(u\cdot\partial_t\phi + \nu u\cdot\Delta\phi + u\cdot(u\cdot\nabla\phi)\Big)dxdt = 0 \tag{6.2}$$

and require that the initial datum u_0 is attained in a certain sense (e.g. $\|u(t)-u_0\|_{L^2}\to 0$ as $t\to 0^+$).

As it turns out, using the Helmholtz-Hodge decomposition, one may recover a pressure field p from a weak solution u of (6.1) with the formula (1.43), under suitable integrability conditions on u.

The divergence free-condition also may be tested against a scalar test function $\phi\in C_0^\infty(\Omega)$ to yield

$$\int_\Omega u\cdot\nabla\phi dx = 0. \tag{6.3}$$

We will henceforth say that u *is divergence-free in the sense of distributions* if (6.3) holds for any scalar $\phi\in C_0^\infty(\Omega)$.

Definition 6.1 (Weak solution of Navier-Stokes)**.** Let $u_0\in L^2_\sigma$. We say that a vector field

$$u\in L^\infty_{\text{loc}}(0,\infty;L^2_\sigma)\cap L^2_{\text{loc}}(0,\infty;\dot{H}^1_\sigma) \tag{6.4}$$

is a weak solution of the Cauchy problem for the Navier-Stokes equations (2.1a) if u is divergence-free in the sense of distributions (i.e. (6.3) holds for any $\phi\in C_0^\infty(\Omega)$), u solves the Navier-Stokes equation in the sense of distributions (i.e. (6.1) holds for any divergence free $\phi\in C_0^\infty([0,\infty)\times\Omega)$), and u attains the initial value u_0 in an L^2 sense, i.e. $\lim_{t\to 0^+}\|u(t)-u_0\|_{L^2} = 0$.

The main result of this section is:

Theorem 6.2 (Leray [Ler34], Hopf [Hop51])**.** *Let $u_0\in L^2_\sigma$. Then there exists at least one global in time weak solution of the Navier-Stokes equation with initial datum u_0, and moreover we the energy inequality*

$$\|u(t)\|^2_{L^2} + 2\nu\int_0^t\|\nabla u(\tau)\|^2_{L^2}d\tau \le \|u_0\|^2_{L^2} \tag{6.5}$$

for all $t>0$, and we have

$$\partial_t u\in L^{4/3}_{\text{loc}}(0,\infty;\dot{H}^{-1}) \tag{6.6}$$

in three dimensions and

$$\partial_t u\in L^2_{\text{loc}}(0,\infty;\dot{H}^{-1}) \tag{6.7}$$

in two dimensions.

The proof of proof of Theorem 6.2 consists of construction a sequence of approximate solutions, and showing that this sequence has a converging subsequence in the energy space (6.4), which is a weak-solution of Navier-Stokes that obeys (6.5) and the stated regularity properties for $\partial_t u$. The difference in the proofs for the whole space $\mathbb{R}^d$ and the periodic domain $\mathbb{T}^d$ lies in the way the Navier-Stokes equations are approximated: for the whole space we use mollification, while for the periodic box we use a Galerkin approximation. The passage to the limit is then the same (albeit on the whole space the convergence is uniform on compact subsets), and appeals to a compactness theorem known as the Aubin-Lions compactness theorem, which we state next.

Theorem 6.3 (Aubin-Lions compactness, L^p version)**.** *Let $X \subset Y \subset Z$ be seperable, reflexive Banach spaces, such that the embedding $X \subset Y$ is compact , and the embedding $Y \subset Z$ is continuous. Let $T > 0$, and assume that we have a sequence of functions $\{u_n\}_{n\geq 1}$ such that*

$$\begin{aligned} &\{u_n\} \text{ is uniformly bounded in } L^p(0,T;X)\\ &\{\partial_t u_n\} \text{ is uniformly bounded in } L^q(0,T;Z) \end{aligned}$$

where $p, q > 1$. Then the sequence $\{u_n\}$ is (strongly) precompact in $L^p(0,T;Y)$.

The strong convergence guaranteed by Theorem 6.3 will be required in order to pass to the limit in the nonlinear term of (6.1). The role of the spaces X, Y, and Z are played by $X = \dot{H}^1$ (here $x \in \mathbb{T}^d$ or an arbitrary compact set $K \subset \mathbb{R}^d$), $Y = L^2$, $Z = H^{-1}$, $p = 2$, and $q = 4/3$ or 2. The proof of Theorem 6.3 is given in Appendix C below.

6.1 Existence of weak solutions on the whole space via mollification

In this section we sketch the proof of Theorem 6.2 for the case of the whole space $\mathbb{R}^d$. We consider a slightly different variant of mollifying the Navier-Stokes equations, defined by

$$\partial_t u^\varepsilon - \nu\Delta u^\varepsilon + \mathbb{P}(\mathcal{J}_\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon) = 0, \qquad u^\varepsilon|_{t=0} = \mathcal{J}_\varepsilon u_0. \tag{6.8}$$

Using that $\mathbb{P}$ is bounded on L^2 and that the operator norm of $\nabla^r e^{t\nu\Delta}\colon L^2 \to L^2$ is bounded by $C/(\nu t)^{r/2}$ for some constant $C = C_{d,r}$, one may show that there exists

$$T = T_\varepsilon = T_\varepsilon(\varepsilon, \|u_0\|_{L^2}, \nu, d) > 0$$

such that the map

$$u(t) \mapsto \Phi_\varepsilon(u)(t) = e^{\nu t\Delta}\mathcal{J}_\varepsilon u_0 + \int_0^t e^{\nu(t-s)\Delta}\mathbb{P}\nabla\cdot(\mathcal{J}_\varepsilon u(\tau)\otimes u(\tau))d\tau$$

is a contraction on the Banach space

$$X_T = L^\infty(0,T;\bar{B}_{L^2_\sigma}(2\|u_0\|_{L^2})).$$

Using the contraction mapping principle, it then follows that there exists a unique fixed point

$$u^\varepsilon \in X_{T_\varepsilon}$$

which thus solves (6.8) in the sense of distributions. In fact, one may choose T_ε as above possibly smaller (still depending on the same quantities), such that we have

$$\sup_{t\in[0,T_\varepsilon]} \sqrt{t}\|\nabla u^\varepsilon(t)\|_{L^2} + \sup_{t\in[0,T_\varepsilon]} t\|\Delta u^\varepsilon(t)\|_{L^2} \leq C\|u_0\|_{L^2}$$

for some $C = C(\varepsilon, \nu) > 0$. Moreover, from the mild formulation of (6.8) it is not hard to check that in fact $u^\varepsilon \in C([0, T_\varepsilon]; L^2)$.

By taking the L^2 inner product of (6.8) with u^ε, and using the smoothness of u^ε that we have obtained earlier, one may show that u^ε obeys the energy equality

$$\|u^\varepsilon(t)\|_{L^2}^2 + 2\nu \int_s^t \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 d\tau = \|u^\varepsilon(s)\|_{L^2}^2 \tag{6.9}$$

for any $0 \le s < t \le T_\varepsilon$. This shows that $\|u^\varepsilon(T_\varepsilon)\|_{L^2} \le \|u_0\|_{L^2}^2$ and thus one may repeat the fixed point argument on intervals of the type $[nT_\varepsilon, (n+1)T_\varepsilon]$, and conclude that the solution u^ε is global in time, that it obeys the energy equality (6.9) for all $0 \le s < t$, and that that we an ε-independent a posteriori bound

$$\{u^\varepsilon\} \qquad \text{is uniformly bounded in} \qquad L^\infty_{\text{loc}}(0,\infty; L^2) \cap L^2_{\text{loc}}(0,\infty; \dot{H}^1). \tag{6.10}$$

Using (6.8), and the Sobolev embeddings, one may further deduce from (6.10) that

$$\{\partial_t u^\varepsilon\} \qquad \text{is uniformly bounded in} \qquad L^{p_d}_{\text{loc}}(0,\infty; \dot{H}^{-1}), \tag{6.11}$$

where

$$p_d = \begin{cases} 2, & d = 2, \\ 4/3, & d = 3. \end{cases}$$

At this stage we may use the Aubin-Lions compactness criterion in Theorem 6.3. Fix an arbitrary $T > 0$ and an arbitrary compact set $K \subset \mathbb{R}^d$, so that the embedding of $H^1(K) \subset L^2(K)$ is compact by Rellich. We let $p = 2$, $q = p_d$, $X = H^1(K)$, $Y = L^2(K)$, and $Z = \dot{H}^{-1}(K)$, and conclude that there exists a limit point

$$u \in L^\infty(0,T; L^2(K)) \cap L^2(0,T; \dot{H}^1(K)), \qquad \partial_t u \in L^{p_d}(0,T; \dot{H}^{-1}(K))$$

and a subsequence $\{\varepsilon_n\} \to 0$ as $n \to \infty$ such that

$$u^{\varepsilon_n} \rightharpoonup^* u \qquad \text{weakly-* in } L^\infty(0,T; L^2(K)), \tag{6.12}$$

$$u^{\varepsilon_n} \rightharpoonup u \qquad \text{weakly in } L^2(0,T; H^1(K)), \tag{6.13}$$

$$u^{\varepsilon_n} \to u \qquad \text{strongly in } L^2(0,T; L^2(K)). \tag{6.14}$$

as $\varepsilon_n \to 0$. In fact, by considering a diagonal argument, one may show that the above convergences hold for all times $T = T_n = n \in \mathbb{Z}_+$, and all compact sets $K = K_n = \bar{B}_{\mathbb{R}^d}(n)$.

It remains to prove that the limiting function u is a weak solution of the Navier-Stokes system (i.e. it solves (6.1), (6.3), and attains the initial datum u_0 in an L^2 sense). We start by noting that for any $T > 0$ we have $u \in C_w(0,T; L^2)$. Indeed, pick a test function $\phi \in C_0^\infty(\mathbb{R}^d)$ and let $K = \text{supp}(\phi)$. Then, from the fact that $u\colon [0,T] \to H^{-1}(K)$ is Hölder continuous, that $L^2(K)$ is dense in $H^{-1}(K)$, and that $u \in L^\infty(0,T; L^2(K))$ it follows that $\int_{\mathbb{R}^d} u(x,t)\phi(x)dx$ is a continuous function of time. The weak divergence free condition is automatic since $u(t) \in L^2_\sigma$ for a.e. $t > 0$, and we have $u \in C_w(0,T; L^2_{\text{loc}})$.

Let $\phi \in C_0^\infty([0,\infty) \times \mathbb{R}^d)$, be divergence-free. To show that u obeys (6.1) we note that u^ε obeys it, i.e.

$$\iint_{[0,\infty)\times\mathbb{R}^d} \Big(u^\varepsilon \cdot \partial_t \phi + \nu u^\varepsilon \cdot \Delta \phi + u^\varepsilon \cdot (u^\varepsilon \cdot \nabla \phi) \Big) dx dt = \int_{\mathbb{R}^d} u_0^\varepsilon \cdot \phi(\cdot, 0)$$

Since $\mathcal{J}_\varepsilon u_0 \to u_0$ in L^2 as $\varepsilon \to 0$, we have that

$$\int_{\mathbb{R}^d} u_0^\varepsilon \cdot \phi(\cdot,0) \to \int_{\mathbb{R}^d} u_0 \cdot \phi(\cdot,0)$$

as $\varepsilon \to 0$. Using (6.12)–(6.13) it follows that

$$\iint_{[0,\infty)\times\mathbb{R}^d} \Big(u^\varepsilon \cdot \partial_t \phi + \nu u^\varepsilon \cdot \Delta\phi\Big) dxdt \to \iint_{[0,\infty)\times\mathbb{R}^d} \Big(u \cdot \partial_t \phi + \nu u \cdot \Delta\phi\Big) dxdt$$

as $\varepsilon \to 0$. The strong convergence of (6.14) is needed only in order to pass to the limit in the nonlinear term. Since the product of a weakly convergent sequence and a strongly convergent sequence necessarily converges weakly, we may immediately show that

$$\iint_{[0,\infty)\times\mathbb{R}^d} u^\varepsilon \cdot (u^\varepsilon \cdot \nabla\phi) dxdt \to \iint_{[0,\infty)\times\mathbb{R}^d} u \cdot (u \cdot \nabla\phi) dxdt$$

as $\varepsilon \to 0$.

Next, we prove that u obeys the energy inequality (6.5). We follow the proof in [LR02]. We consider a test function $\alpha(t) \in C_0^\infty([0,\infty))$. By (6.12)–(6.13) we have that $\alpha u^\varepsilon \rightharpoonup \alpha u$ in $L^2([0,\infty)\times\mathbb{R}^d)$, and $\nabla u^\varepsilon \rightharpoonup \nabla u$ in $L^2([0,\infty)\times\mathbb{R}^d)$. Thus, using properties of weak convergence and the energy equality obeyed by u^ε we obtain that

$$\begin{aligned}
&\iint_{[0,\infty)\times\mathbb{R}^d} |\alpha(t)|^2 |u(x,t)|^2 dxdt + 2\nu \int_0^\infty |\alpha(t)|^2 \left(\int_0^t \int_{\mathbb{R}^d} |\nabla u(x,\tau)|^2 dx d\tau\right) dt \\
&\le \liminf_{\varepsilon\to 0} \iint_{[0,\infty)\times\mathbb{R}^d} |\alpha(t)|^2 |u^\varepsilon(x,t)|^2 dxdt + 2\nu \int_0^\infty |\alpha(t)|^2 \left(\int_0^t \int_{\mathbb{R}^d} |\nabla u^\varepsilon(x,\tau)|^2 dx d\tau\right) dt \\
&= \iint_{[0,\infty)\times\mathbb{R}^d} |\alpha(t)|^2 |u_0^\varepsilon(x)|^2 dxdt \\
&\le \|u_0\|_{L^2}^2 \int_0^\infty |\alpha(t)|^2 dt.
\end{aligned}$$

Let $\theta \in C_0^\infty([0,\infty))$ be such that $\int_0^\infty |\theta(\tau)|^2 d\tau = 1$, fix $t_0 > 0$, and for $\eta > 0$ define

$$\alpha(t) = \frac{1}{\eta^{1/2}} \theta\left(\frac{t-t_0}{\eta}\right).$$

Inserting this test function in the previous energy inequality and passing $\eta \to 0^+$ we obtain

$$\limsup_{\eta\to 0^+} \int_0^\infty \frac{1}{\eta}\theta^2\left(\frac{t-t_0}{\eta}\right) \|u(t)\|_{L^2(\mathbb{R}^d)}^2 dt + 2\nu \int_0^{t_0} \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2.$$

For t_0 in the Lebesgue set of the map $t \mapsto \|u(t)\|_{L^2}^2$, the first term on the left side of the above converges to $\|u(t_0)\|_{L^2}^2$. To extend this convergence from the Lebesgue set to all the points $t_0 \ge 0$, we recall that $u \in C_w([0,\infty); L^2)$. Thus, choosing $t_n \to t_0$ as $n \to \infty$, with t_n in the Lebesgue set of $t \mapsto \|u(t)\|_{L^2}^2$, we have that

$$\|u(t_0)\|_{L^2}^2 + 2\nu \int_0^{t_0} \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \liminf_{n\to\infty} \|u(t_n)\|_{L^2}^2 + 2\nu \int_0^{t_n} \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|u_0\|_{L^2}^2$$

where we used that for the dissipative term $\liminf$ becomes $\lim$.

Lastly, the fact that $\|u(t) - u_0\|_{L^2} \to 0$ as $t \to 0^+$ follows from weak continuity in time, and the fact that $\|u(t)\|_{L^2} \to \|u_0\|_{L^2}$. The latter follows since using properties of weak convergence we have $\|u_0\|_{L^2} \le \liminf_{t\to 0^+} \|u(t)\|_{L^2}$, while from the energy inequality we have that $\limsup_{t\to 0^+} \|u(t)\|_{L^2} \le \|u_0\|_{L^2}$. This concludes the proof of the Leray existence theorem for the case of the whole space.

6.2 Existence of weak solutions on the periodic box via Galerkin

In this section we give the proof of Theorem 6.2 for the periodic domain $\mathbb{T}^d$. Here, the classical way to approximate the Navier-Stokes equations on the periodic box (or for that matter on a bounded domain) is to use a so-called *Galerkin approximation*. The idea here is that the eigenfunctions of the Stokes operator $-\nu\mathbb{P}\Delta$ (on the periodic box $\mathbb{T}^d$ the Leray projector commutes with the negative Laplacian) form an orthonormal basis for L^2_σ (recall (1.51)). For the periodic domain this orthonormal basis is given by $\phi_k(x) = a_k \exp(ik\cdot x)$, where $|a_k| = 1$, and $k \cdot a_k = 0$. Note that for $d = 2$ we have one independent vector a_k , while for $d = 3$, we have two linearly independent vectors a_k. The corresponding eigenvalues are $\nu|k|^2$ with $k \in \mathbb{Z}^d \setminus \{0\}$, which may be ordered linearly, and gives a natural way to approximate the Navier-Stokes flow.

For this purpose, let $m > 0$, and define $\mathbb{P}_m \colon L^2 \to L^2_\sigma$ as follows. For an L^2 vector field φ, we write its Fourier series expansion

$$\varphi(x) = \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}(k) e^{ik\cdot x}$$

and define

$$\mathbb{P}_m\varphi(x) = \sum_{0<|k|\leq m} \left(\widehat{\varphi}(k) - \left(\widehat{\varphi}(k)\cdot\frac{k}{|k|}\right)\frac{k}{|k|}\right) e^{ik\cdot x}. \tag{6.15}$$

In particular, it is clear that $\mathbb{P}_m\varphi \in L^2_\sigma$. Moreover, $\mathbb{P}_m$ is given by a Fourier multiplier, so it commutes with derivatives, and with convolution operators.

With this definition, the mth Galerkin truncation of the Navier-Stokes equations is given by the system

$$\partial_t u_m - \nu\Delta u_m + \mathbb{P}_m(u_m\cdot\nabla u_m) = 0, \qquad u_m|_{t=0} = P_m u_0. \tag{6.16}$$

In view of the aforementioned properties of the projection operator $\mathbb{P}_m$ defined in (6.15), we note that (6.16) denotes an evolution on a finite dimensional Banach space $\mathbb{P}_m L^2 \subset L^2_\sigma$. Also, note that for $\varphi \in \mathbb{P}_m L^2$, we have $\mathbb{P}_m\varphi = \varphi$, and the $\mathbb{P}_m$ is self-adjoint with respect to the $\dot{H}^s$ inner product, for all $s \in \mathbb{R}$.

6.2.1 Existence of solutions and uniform estimates for the Galerkin approximation

As mentioned above, (6.15) is a finite dimensional (this dimension equals the number of lattice points on $\mathbb{Z}^d$ inside the ball of radius m in $\mathbb{R}^d$ and is $\approx m^d$) system of autonomous quadratic ODEs. The existence of a unique solution u_m on a maximal time interval T_m is guaranteed by the multi-dimensional Picard-Lindelöf existence and uniqueness theorem for ODEs (see Theorem A.2). Since this system is finite dimensional (and thus all norms are equivalent) the maximal time of existence T_m is characterized by

$$\limsup_{t\to T_m^-} \|u_m(t)\|^2_{L^2} = \limsup_{t\to T_m^-} (2\pi)^d \sum_{0<|k|\leq m} |\widehat{u_m}(k,t)|^2 = \infty.$$

At this stage we note that the Galerkin solution u_m obey a good energy estimate. Indeed, taking an L^2 inner product of (6.16) with u_m, and using that $\mathbb{P}_m$ is self adjoint on L^2, and the identity on $\mathbb{P}_m L^2$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2} + \nu\|\nabla u_m\|^2_{L^2} = -\int_{\mathbb{T}^d} (u_m\cdot\nabla u_m)\cdot u_m dx = -\frac{1}{2}\int_{\mathbb{T}^d} u_m\cdot\nabla(|u_m|^2)dx = 0. \tag{6.17}$$

Integrating (6.17) on $(0,t)$ we obtain the energy equality

$$\|u_m(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla u_m(\tau)\|^2_{L^2} d\tau = \|\mathbb{P}_m u_0\|^2_{L^2} \leq \|u_0\|^2_{L^2} \tag{6.18}$$

for all $t > 0$. On the other hand, using the Póincare inequality

$$\|\nabla\varphi\|^2_{L^2(\mathbb{T}^d)} \geq \|\varphi\|^2_{L^2(\mathbb{T}^d)} \qquad \text{for all} \qquad \varphi \in H^1_\sigma \tag{6.19}$$

we deduce from (6.17) that

$$\|u_m(t)\|^2_{L^2} \leq e^{-\nu t}\|u_0\|^2_{L^2} \tag{6.20}$$

for all $t > 0$. It thus follows from (6.18) and (6.20) that the Galerkin sequence obeys

$$\{u_m\}_{m>0} \qquad \text{is uniformly bounded in} \qquad L^\infty([0,\infty); L^2_\sigma) \cap L^2([0,\infty); \dot{H}^1_\sigma) \tag{6.21}$$

with bounds that may of course depend on ν.

At this stage we use the Gagliardo-Nirenberg-Sobolev inequality (i.e., the embedding of H^1 in L^4 for $d \leq 4$) and the Fourier characterization of Sobolev spaces to obtain

$$\|\nu\Delta u_m(t)\|_{\dot{H}^{-1}} = \nu\|\nabla u_m(t)\|_{L^2}$$

and

$$\begin{aligned}
\|\mathbb{P}_m(u_m\cdot\nabla u_m)\|_{\dot{H}^{-1}} &= \|\mathbb{P}_m\nabla\cdot(u_m\otimes u_m)\|_{\dot{H}^{-1}} \\
&= \|u_m\otimes u_m\|_{L^2} \\
&\leq \|u_m\|^2_{L^4} \\
&\leq C\|u_m\|^{2-d/2}_{L^2}\|\nabla u_m\|^{d/2}_{L^2}.
\end{aligned}$$

From the above two estimates and the uniform bound (6.21) we obtain that

$$\{\partial_t u_m\}_{m>0} \qquad \text{is uniformly bounded in} \qquad L^{p_d}_{\text{loc}}([0,\infty); \dot{H}^{-1}_\sigma) \tag{6.22}$$

where $p_d = 2$ if $d = 2$, and $p_d = 4/3$ if $d = 3$.

6.2.2 The Galerkin sequence has a convergent subsequence which is a weak solution of Navier-Stokes

In view of (6.21)–(6.22) we may now apply the Aubin-Lions compactness theorem (see Theorem 6.3 above) to the sequence of Galerkin truncations, and a diagonal argument for the sequence of times $T = T_n = n \geq 1$, to conclude that there exists a limit point

$$u \in L^\infty_{\text{loc}}([0,\infty); L^2_\sigma) \cap L^2_{\text{loc}}([0,\infty); \dot{H}^1_\sigma), \qquad \partial_t u \in L^{p_d}_{\text{loc}}([0,\infty); \dot{H}^{-1}_\sigma) \tag{6.23}$$

and a subsequence $m_k \to \infty$ such that

$$u^{m_k} \rightharpoonup u \qquad \text{in} \qquad L^2_{\text{loc}}([0,\infty); \dot{H}^1_\sigma) \tag{6.24}$$

$$u^{m_k} \rightharpoonup^* u \qquad \text{in} \qquad L^\infty_{\text{loc}}([0,\infty); L^2_\sigma) \tag{6.25}$$

$$\partial_t u^{m_k} \rightharpoonup \partial_t u \qquad \text{in} \qquad L^{p_d}_{\text{loc}}([0,\infty); \dot{H}^{-1}_\sigma) \tag{6.26}$$

$$u^{m_k} \to u \qquad \text{in} \qquad L^2_{\text{loc}}([0,\infty); L^2_\sigma) \tag{6.27}$$

as $k \to \infty$. Note that the limit point u may not be unique.

Moreover, note that in view of (6.23) and the fundamental theorem of calculus (with respect to t), which gives the embedding

$$C^{\alpha_p}(\mathbb{R}) \subset W^{1,p}(\mathbb{R}), \qquad \text{for } p > 1, \text{ where } \alpha_p = \frac{p-1}{p},$$

we have that

$$u \in C^{\alpha_d}_{\text{loc}}([0,\infty); \dot{H}^{-1}_\sigma) \tag{6.28}$$

where $\alpha_d = 1/2$ if $d = 2$, and $\alpha_d = 1/4$ if $d = 3$. Note that because H^1_σ is dense in L^2_σ, and since u is bounded in time as a function with values in L^2_σ, it follows from (6.28) a standard density argument that

$$u \in C_w([0,\infty); L^2_\sigma),$$

i.e. that u is weakly continuous with values in L^2_σ.

We next show that any limit point u as obtained above solves the Navier-Stokes equations in the sense of distributions. First, since $u \in L^2_\sigma$, we have that u is divergence free in the sense of distributions. Next, fix $t > 0$ and consider a test function $\phi \in C^\infty([0,t] \times \mathbb{T}^d)$, which is divergence-free. Taking the L^2 inner product of (6.16) with ϕ and integrating over $[0,t]$, we arrive at

$$\int_{\mathbb{T}^d} u_m(t) \cdot \phi dx + \nu \int_0^t \int_{\mathbb{T}^d} \nabla u_m(\tau) \cdot \nabla \phi dx d\tau - \int_0^t \int_{\mathbb{T}^d} u_m(\tau) \cdot \nabla \phi \cdot u_m(\tau) dx d\tau = \int_{\mathbb{T}^d} \mathbb{P}_m u_0 \cdot \phi dx. \tag{6.29}$$

Since $\|u_0 - \mathbb{P}_m u_0\|_{L^2} \to 0$ as $m \to \infty$, we immediately have

$$\int_{\mathbb{T}^d} \mathbb{P}_m u_0 \cdot \phi dx \to \int_{\mathbb{T}^d} u_0 \cdot \phi dx$$

as $m \to \infty$. In view of (6.24), we also obtain that

$$\nu \int_0^t \int_{\mathbb{T}^d} \nabla u_m(\tau) \cdot \nabla \phi dx d\tau \to \nu \int_0^t \int_{\mathbb{T}^d} \nabla u(\tau) \cdot \nabla \phi dx d\tau$$

as $m \to \infty$. At this stage we shall use the strong convergence (the only time). We write

$$\begin{aligned}
&\int_0^t \int_{\mathbb{T}^d} u_m(\tau) \cdot \nabla \phi \cdot u_m(\tau) dx d\tau - \int_0^t \int_{\mathbb{T}^d} u(\tau) \cdot \nabla \phi \cdot u(\tau) dx d\tau \\
&= \int_0^t \int_{\mathbb{T}^d} (u_m(\tau) - u(\tau)) \cdot \nabla \phi \cdot (u_m(\tau) - u(\tau)) dx d\tau + \int_0^t \int_{\mathbb{T}^d} u(\tau) \cdot \nabla \phi \cdot (u_m(\tau) - u(\tau)) dx d\tau \\
&\quad + \int_0^t \int_{\mathbb{T}^d} (u_m(\tau) - u(\tau)) \cdot \nabla \phi \cdot u(\tau) dx d\tau.
\end{aligned}$$

The last two terms in the above converge to 0 as $m \to \infty$ since $\nabla \phi \in L^{3/2}_x \subset L^2_x$, $u \in L^2_t L^6_x \subset L^2_t H^1_x$ (both for $d = 2$ and $d = 3$), and cf. (6.24) we have that u_m converges to u weakly in $L^2_t L^6_x \subset L^2_t H^1_x$ (both for $d = 2$ and $d = 3$). The strong convergence is used for the first term, which is bounded using the Sobolev embedding as

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{T}^d} (u_m(\tau) - u(\tau)) \cdot \nabla \phi \cdot (u_m(\tau) - u(\tau)) dx d\tau \right| \\
&\quad \leq \|u_m - u\|^2_{L^2(0,t;L^4_x)} \|\nabla \phi\|_{L^2_x} \\
&\quad \leq \|u_m - u\|^{2-d/2}_{L^2(0,t;L^2_x)} \left(\|\nabla u_m\|^{d/2}_{L^2(0,t;L^2_x)} + \|\nabla u\|^{d/2}_{L^2(0,t;L^2_x)} \right) \|\nabla \phi\|_{L^2_x}
\end{aligned}$$

which converges to 0 as $m \to \infty$ in view of (6.27). Lastly, from (6.24) we conclude that for a.e. $t \geq 0$ (in the Lebesgue set of the map $t \mapsto \|u(t)\|_{L^2}$) we have

$$\int_{\mathbb{T}^d} u_m(t) \cdot \phi dx \to \int_{\mathbb{T}^d} u(t) \cdot \phi dx.$$

Now, to get rid of this possibly bad zero measure set in time, we recall that u_m are uniformly equicontinuous with values in H_σ^{-1}, and that u is weakly continuous with values in L^2. This proves that for all $t \geq 0$ we have that

$$\int_{\mathbb{T}^d} u(t) \cdot \phi dx + \nu \int_0^t \int_{\mathbb{T}^d} \nabla u(\tau) \cdot \nabla \phi dx d\tau - \int_0^t \int_{\mathbb{T}^d} u(\tau) \cdot \nabla \phi \cdot u(\tau) dx d\tau = \int_{\mathbb{T}^d} u_0 \cdot \phi dx \tag{6.30}$$

holds for all $\phi \in H_\sigma^1$. Thus, since $t > 0$ was arbitrary, we have shown that u solves the weak formulation of the Navier-Stokes equations, in the sense of (6.1).

6.2.3 The limiting solution obeys the energy inequality

Using (6.27) and the fact that if $x_n \rightharpoonup x$ in a Hilbert space, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$, we have

$$\int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \liminf_{m\to\infty} \int_0^t \|\nabla u_m(\tau)\|_{L^2}^2 d\tau.$$

On the other hand, using (6.25), we have that we have that

$$\|u(t)\|_{L^2}^2 \leq \limsup_{m\to\infty} \|u_m(t)\|_{L^2}^2.$$

Thus, taking the $\limsup_{m\to\infty}$ on the right side of (6.18), and using that $\limsup(a_n + b_n) \geq \limsup a_n + \liminf b_n$, we arrive at

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2$$

for all $t \geq 0$, thereby proving (6.5).

Exercise 6.4. With a bit more care, show that the energy inequality (6.5) holds on $[t_0, t]$, for a.e. $t_0 \geq 0$ and all $t \geq t_0$.

Lastly we prove that

$$\|u(t) - u_0\|_{L^2} \to 0 \qquad \text{as} \qquad t \to 0^+.$$

From the weak continuity in time of u with values in L^2, we obtain from one hand that

$$\|u_0\|_{L^2} \leq \liminf_{t\to 0^+} \|u(t)\|_{L^2}.$$

On the other hand, from the energy inequality (6.5) we have that

$$\limsup_{t\to 0^+} \|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$$

which concludes the proof.

6.3 The uniqueness of weak solutions in 2D

Theorem 6.5 (Uniqueness of Leray-Hopf weak solutions in 2D)**.** *Assume u and v are two Leray-Hopf weak soltutions of the Navier-Stokes equations, with initial datum $u_0 = v_0$. Then $u \equiv v$.*

Proof of Theorem 6.5. Consider $w = u - v$. Then w solves

$$\partial_t w - \nu \Delta w + u \cdot \nabla w + w \cdot \nabla v + \nabla q = 0, \qquad w|_{t=0} = 0$$

in the sense of distributions, for some pressure q. In fact, in view of the integrability properties of u, v and thus w, we have that w solves the above equation in $L^2_{\mathrm{loc}}([0,\infty); \dot{H}^{-1})$ (each term lies in this space, and $C_0^\infty([0,\infty) \times \mathbb{T}^2)$ is dense in this space). In particular, this justifies taking an L^2 inner product with a function in $L^2_t \dot{H}^1_x$, more precisely, w itself. We then obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \nu \|\nabla w\|_{L^2}^2 &= -\int (w \cdot \nabla v) \cdot w dx = \int w \cdot (v \cdot \nabla w) dx \\
&\leq \|\nabla w\|_{L^2} \|w\|_{L^4} \|v\|_{L^4} \\
&\leq C \|\nabla w\|_{L^2}^{3/2} \|w\|_{L^2}^{1/2} \|v\|_{L^2}^{1/2} \|\nabla v\|_{L^2}^{1/2} \\
&\leq \frac{\nu}{2} \|\nabla w\|_{L^2}^2 + \frac{C}{\nu^3} \|w\|_{L^2}^2 \left(\|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2\right)
\end{aligned}$$

where we have used that in two space dimensions the Ladyzhenskaya inequality holds

$$\|f\|_{L^4} \leq C \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2}$$

if $\int_{\mathbb{T}^2} f dx = 0$. On the other hand from the energy inequality for v we have that

$$2\nu \int_0^t \|v(\tau)\|_{L^2}^2 \|\nabla v(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^4$$

and thus applying the Grönwall inequality to the function $\|w(t)\|_{L^2}^2$, which starts off with $\|w_0\|_{L^2}^2 = 0$, shows that $w = 0$. □

Remark 6.6. Whether weak solutions are unique in 3D remains an outstanding open problem, for which the Clay foundation has set a reward.

6.4 Strong solutions

Definition 6.7 (Strong solution)**.** A weak solution u of the Navier-Stokes equations is called *strong* on $[0,T]$ if one of the following estimates hold:

(i) For $d = 2$, and $u_0 \in L^2_\sigma$, we have

$$u \in L^\infty_{\mathrm{loc}}(0,T,H^1_\sigma) \cap L^2_{\mathrm{loc}}(0,T;H^2_\sigma).$$

(ii) For $d \in \{2,3\}$ and $u_0 \in H^1_\sigma$, we have

$$u \in C_{\mathrm{w}}(0,T,H^1_\sigma) \cap L^2(0,T;H^2_\sigma).$$

Theorem 6.8 (Global existence of strong solutions in 2D)**.** *Assume $u_0 \in L^2_\sigma(\mathbb{T}^2)$. Given $T > 0$ arbitrary, there exists a strong solution u of the Navier-Stokes equation on $[0,T]$. Moreover, we have the estimate*

$$\sup_{0 < t \leq \min\{T,(C_0\nu)^{-1}\}} \nu t \|u(t)\|_{\dot{H}^1}^2 + \sup_{\min\{T,(C_0\nu)^{-1}\} \leq t \leq T} \|u(t)\|_{\dot{H}^1}^2 \leq M_0^2$$

where $C_0 > 0$ is an absolute constant, and $M_0 = M_0(\nu, \|u_0\|_{L^2}) > 0$ may be computed explicitly, and is independent of T. If additionally we have have that $u_0 \in H^1_\sigma$, then the strong solution obeys

$$\sup_{0 < t \leq T} \|u(t)\|_{\dot{H}^1}^2 + \nu \int_0^T \|u(\tau)\|_{\dot{H}^2} d\tau \leq M_1^2(\nu^{-1} + T)$$

where $M_1 = M_1(\nu, \|u_0\|_{L^2}) > 0$ may be computed explicitly, and is independent of T.

Theorem 6.9 (Local existence of strong solutions in 3D)**.** *Let* $u_0 \in H^1_\sigma(\mathbb{T}^3)$ *and fix* $\nu > 0$*. There exits* $T_0 = T_0(\nu, \|u_0\|_{H^1}) > 0$ *and a strong solution* u *of the Navier-Stokes equation on* $[0, T]$*, which obeys the estimate*

$$\sup_{t\in[0,T_0]} \|u(t)\|^2_{H^1} \leq 2\|u_0\|^2_{H^1} + C\nu^2$$

for some constant $C > 0$*.*

Theorem 6.10 (Global existence of strong solutions in 3D for small datum)**.** *There exists* $\varepsilon > 0$*, which depends only on the diameter of the torus, such that if* $\|u_0\|_{\dot{H}^1} \leq \varepsilon\nu$*, then there exists a global in time strong solution of the Navier-Stokes equation, which obeys* $\|u(t)\|_{\dot{H}^1} \leq 8\varepsilon\nu$ *for all* $t > 0$*.*

Theorem 6.11 (Strong solutions are unique in 3D)**.** *Consider two Leray-Hopf weak solutions* u, v *of the 3D Navier-Stokes equation, with* $u_0 = v_0 \in H^1_\sigma$*. If* v *is a strong solution, then* $u \equiv v$*.*

Proof of Theorem 6.11. The proof mimics that of Theorem 6.5. Indeed, using that in three space dimensions the Gagliardo-Nirenberg inequality yields

$$\|f\|_{L^3} \leq C\|f\|^{1/2}_{L^2}\|\nabla f\|^{1/2}_{L^2}$$

when $\int_{\mathbb{T}^2} f dx = 0$, we obtain that the difference of the two solutions $w = u - v$ obeys

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} + \nu\|\nabla w\|^2_{L^2} &\leq \|w\|^2_{L^3}\|\nabla v\|_{L^3} \\ &\leq C\|w\|_{L^2}\|\nabla w\|_{L^2}\|\nabla v\|^{1/2}_{L^2}\|\Delta v\|^{1/2}_{L^2} \\ &\leq \frac{\nu}{2}\|\nabla w\|^2_{L^2} + \frac{C}{\nu}\|w\|^2_{L^2}\left(\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}\right).\end{aligned}$$

Since $\|w_0\|_{L^2} = 0$, and by the assumptions on v being a strong solution we have that

$$\int_0^t \|\nabla v(\tau)\|_{L^2}\|\Delta v(\tau)\|_{L^2} d\tau < \infty,$$

we conclude from the Grönwall inequality that $w \equiv 0$, which concludes the proof. □

Remark 6.12. Note that the integrality assumptions on a "strong" solution are more than what is required to obtain uniqueness. The sharp condition turns out to be finiteness of the the scaling invariant (with respect to the natural scaling of the Navier-Stokes equations) norm

$$L^p_t L^q_x \qquad \text{where} \qquad \frac{2}{p} + \frac{3}{q} = 1.$$

The proof in the above mentioned critical case is more delicate, and we postpone it to Section 8. When the above equality is replaced by < 1, the proof is rather direct, and we present it below, following the presentation of [Ser62a].

Theorem 6.13 (Subcritical Prodi-Serrin implies solutions are strong)**.** *Let* $u \in L^p_t L^q_x$*, with* $2/p + 3/q < 1$*, be a Leray-Hopf weak solution of the Navier-Stokes equations on* $[0, T]$*. If* $u_0 \in H^1_\sigma$*, then* u *is in fact a strong solution on* $[0, T]$*. If merely* $u_0 \in L^2_\sigma$*, then* u *is a strong solution on* $[t_0, T]$ *for any* $t_0 > 0$*.*

Proof of Theorem 6.13. We consider first the case of $\omega_0 = \nabla \times u_0 \in L^2$. Consider the vorticity formulation of the Navier-Stokes equation

$$\partial_t \omega - \nu \Delta \omega + u \cdot \nabla \omega = \omega \cdot \nabla u.$$

Taking an L^2 inner product with ω and using that $\nabla \cdot u = 0$, we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 &= \int_{\mathbb{R}^d} (\omega \cdot \nabla u) \cdot \omega dx \\ &= - \int_{\mathbb{R}^d} (\omega \cdot \nabla \omega) \cdot u dx \\ &\leq \|\nabla \omega\|_{L^2} \|u\|_{L^q} \|\omega\|_{L^{2q/(q-2)}} \end{aligned}$$

where $q \geq 2$. We now use the 3D Gagliardo-Nirenberg-Sobolev inequality

$$\|\omega\|_{L^r} \leq C_r \|\omega\|_{L^2}^{1+3/r-3/2} \|\nabla \omega\|_{L^2}^{3/2-3/r}$$

which holds for $r \in [2, 6]$. Setting $r = 2q/(q-2)$, yields the restriction $q \geq 3$. Combining the above estimates, we see that

$$\frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 \leq C_q \|u\|_{L^q} \|\omega\|_{L^2}^{1-3/q} \|\nabla \omega\|_{L^2}^{1+3/q}$$

where the constant C_q is finite in the aforementioned range of q.

At this stage we encounter two cases. When $d = 3 = q$, we obtain the inequality

$$\frac{d}{dt} \|\omega\|_{L^2}^2 + \nu \|\nabla \omega\|_{L^2}^2 \leq C_3 \|u\|_{L^q} \|\nabla \omega\|_{L^2}^2$$

which shows that if

$$\|u\|_{L^\infty(0,T;L^3)} \leq \varepsilon = \frac{\nu}{2C_3}$$

then

$$\frac{d}{dt} \|\omega\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \omega\|_{L^2}^2 \leq 0$$

on $[0, T]$, and thus $u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2)$, being a strong solution on this time interval.

The other case is $q > 3$. Here we note that $1 + 3/q < 2$, and thus from the $\varepsilon-$Young inequality we conclude that

$$\frac{d}{dt} \|\omega\|_{L^2}^2 + \frac{\nu}{2} \|\nabla \omega\|_{L^2}^2 \leq C_{\nu,q} \|u\|_{L^q}^{4q/(q-3)} \|\omega\|_{L^2}^2.$$

Therefore, assuming that

$$\|u\|_{L^p(0,T;L^q)} < \infty, \qquad \text{where} \qquad p = \frac{4q}{q-3}$$

we conclude from the Grönwall inequality that $\omega \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, so that u is a strong solution on this time interval. We note that in this case the pair (p, q) obeys

$$\frac{2}{p} + \frac{3}{q} = \frac{q-3}{2q} + \frac{3}{q} = \frac{q+3}{2q} < 1$$

since $q > 3$.

The case when $u_0 \in L^2_\sigma$, we use that for a.e. $t_0 > 0$, from the energy inequality, we have that $\|\nabla u(t_0)\|_{L^2} < \infty$. A quantitative bounds may be obtained from the Chebyshev inequality, or the estimate $\sup_{t \in [0,T]} \sqrt{t} \|\nabla u(t)\|_{L^2} \leq \|u_0\|_{L^2}$, which holds if T is sufficiently small. □