7 Semilinear methods for Navier-Stokes and Fujita-Kato

The weak solutions of Leray are nice because they are global in time and satisfy the energy inequality. However, they are not known to be unique and they lack a number of other nice properties. There is yet another notion of solution between strong and weak which is quite natural. We will consider just the case of $\mathbb{R}^3$; the case of bounded domains is discussed briefly below.

7.1 Mild formulation

Recall the Leray projection $\mathbb{P}$ defined above in §1.10 and apply to (2.1a),

$$\partial_t u + \mathbb{P}\nabla \cdot (u \otimes u) = \nu \mathbb{P}\Delta u$$
$$\nabla \cdot u = 0.$$

In the absence of boundaries, $\mathbb{P}$ is a Fourier multiplier and hence commutes with Δ and we derive

$$\partial_t u + \mathbb{P}\nabla \cdot (u \otimes u) = \nu \Delta u.$$

Since this is semi-linear, it makes sense to use Duhamel's formula and consider solutions of the form

$$u(t) = e^{\nu t\Delta} u_0 - \int_0^t e^{\nu(t-\tau)\Delta} \mathbb{P}\nabla \cdot (u \otimes u)\, d\tau. \qquad (7.1)$$

eq:mild

Here, we are using $e^{\nu t\Delta}$ to denote the semigroup for the heat equation in $\mathbb{R}^3$. Solutions of this general kind are normally called *mild solutions*; this kind of solution usually only make sense for semi-linear equations. For this reason, it can sometimes be problematic to use this kind of representation for inviscid limits, where the equations become quasi-linear in the limit. If there are boundaries, then we do not quite get this, we instead have to leave the $\mathbb{P}\Delta$ and we will need to use this operator, supplemented with the proper boundary conditions (known as the *Stokes operator*), as the linear propagator for the semi-group. This is harder; we will instead focus on the cases without boundary where we just have the heat equation. Below, we denote the bilinear operator

$$B(f, g) = \mathbb{P}\nabla \cdot (f \otimes g).$$

Recall the scaling symmetry of 3D Navier-Stokes: if $u(t, x)$ is a solution to Navier-Stokes on the interval $[0, T)$ then for any λ,

$$u_\lambda(t, x) = \lambda^{-1} u(t\lambda^{-2}, x\lambda^{-1}) \text{ solves Navier-Stokes on the interval } [0, T\lambda^2).$$

In the context of local well-posedness, we roughly classify spaces in the following way: for initial data $f(x)$ write $f_\lambda = \lambda^{-1} f(x\lambda^{-1})$ and for $f \in C_0^\infty$ classify spaces as

$$\lim_{\lambda \to 0} \|f_\lambda\|_X = \infty \Rightarrow \text{subcritical}$$
$$\lim_{\lambda \to 0} \|f_\lambda\|_X \approx \|f\|_X \Rightarrow \text{critical}$$
$$\lim_{\lambda \to 0} \|f_\lambda\|_X = 0 \Rightarrow \text{supercritical}.$$

Let us briefly discuss *why* we are classifying spaces in this way. *Heuristically*, in subcritical spaces, if information goes to small scales in the scaling consistent with Navier-Stokes, then the norm blows up: it would cost Navier-Stokes an infinite amount of X to send information to high frequencies. For supercritical spaces it would require no cost. For critical spaces it would require a finite but non-zero amount. Another good question is: *why* do we care about high frequencies? Why don't we think about what happens as $\lambda \to \infty$? Well, that is because the nonlinearity involves derivatives of u. If low frequencies for some reason dominated the nonlinear effects, we may well be classifying a space by $\lambda \to \infty$.

Another interesting note: suppose that the maximal existence time of a solution depended only on $\|u(0)\|_X$ (the norm of the initial data). If the norm is subcritical, then sending $\lambda \to 0$ increases the norm and also decreases the existence time. If the norm is supercritical however, we get something a bit pathological: as the initial norm goes to zero, so does the existence time! This basically suggests that if there is a single solution which blows up in finite time then the solution map of the PDE will be (rather badly) discontinuous at zero in this topology, since by re-scaling that one blow-up solution, there will be sequence of solutions which converges to zero but which blow up arbitrarily fast. If the norm is critical, then we reach a kind of contradiction: it only makes sense for the blow-up time to depend only the norm if the solution is global in time. We will see that generally, solutions with small initial critical norm will be global in time, whereas often the existence time of larger solutions depends on more detailed information about the initial data.

7.2 Local well-posedness in $\dot{H}^{1/2}$

When one is dealing with a scale-invariant system (that is, one which has a scaling symmetry), it is often optimal to consider local well-posedness in a scale-invariant critical space: one which satisfies, for $f_\lambda = \lambda^{-1} f(x\lambda^{-1})$ and for $f \in C_0^\infty$,

$$\|f_\lambda\|_X = \|f\|_X .$$

The two most obvious critical spaces are the pair $\dot{H}^{1/2} \subset L^3$ (see Proposition G.5). We will first consider the local well-posedness of solutions with $u_0 \in \dot{H}^{1/2}$ where things are a little cleaner. One would like to prove the existence of a solution which satisfies at least $u(t) \in L_t^\infty \dot{H}_x^{1/2} \cap L_t^2 \dot{H}_x^{3/2}$ (the latter from the dissipation). We will do so via the contraction mapping principle, however, it will be advantageous to use the norm $L_t^4 \dot{H}_x^1$ for the primary argument. This norm is slightly weaker in the sense that by Hölder and Sobolev interpolation (which is just Cauchy-Schwarz),

$$\|u\|_{L_t^4 \dot{H}_x^1} \leq \|u\|_{L_t^\infty \dot{H}_x^{1/2}}^{1/2} \|u\|_{L_t^2 \dot{H}_x^{3/2}}^{1/2} .$$

The fact that the norm is L_t^4 rather than L_t^∞ will be important, since this will imply that picking T small will make $L^4(0,T;\dot{H}^1)$ small, even with large data. This will allow us to construct unique, local-in-time solutions from large data. It is important to note that using a contraction mapping principle essentially uses smallness in some way – it is a perturbative argument and does not really make sense out of the perturbative regime. Hence, to use it, we will always need smallness.

Denote

$$\|u\|_{X_T} = \|u\|_{L_t^4(0,T;\dot{H}_x^1)}$$
$$X_T := \left\{ u : \|u\|_{X_T} < \infty \right\}.$$

We will prove the following local-wellposedness theorem.

thm:FKH12 **Theorem 7.1** (Fujita-Kato [FK64]). *Let $u_0 \in \dot{H}_\sigma^{1/2}$. Then, there exists a $0 < T = T(u_0, \nu)$ and a unique solution to (7.1) in the class $C([0,T], \dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ and the solution map $M : \dot{H}^{1/2} \to L_t^4 \dot{H}_x^1$ is locally Lipschitz continuous with respect to the initial data. Moreover, there exists a universal $\epsilon_0 > 0$ such that if $\|u_0\|_{\dot{H}^{1/2}} < \epsilon_0 \nu$, then the mild solution is global in time.*

Remark 7.2. The estimated existence time $T(u_0, \nu)$ depends on the *rate* at which the following limit converges:

$$\lim_{T \to 0^+} \left\| e^{t\Delta} u_0 \right\|_{L^4_t(0,T;H^1_x)} = 0.$$

That it converges to zero at all for $u_0 \in \dot{H}^{1/2}$ is proved below in Lemma 7.3 below.

Proof. For $u_0 \in \dot{H}^{1/2}$ and $\rho \in L^4(0,T;\dot{H}^1)$ (for a T to be chosen below), define the mapping $\Phi[\rho] \to v$ given by

$$v(t) = e^{t\nu\Delta} u_0 - \int_0^t e^{\nu(t-\tau)\Delta} B(\rho,\rho)(\tau) d\tau.$$

It is important to note that we are going to be essentially using $e^{\nu t \Delta} u_0$ as an approximate solution and doing a perturbation argument around it. The goal now is to prove that for $T < T(u_0)$ chosen sufficiently small, Φ defines a contraction on a closed subset of X_T. Then by the contraction mapping principle, Theorem I.1, there exists a unique fixed point, and by definition, this fixed point is a mild solution to the 3D Navier-Stokes equations.

Define for ϵ, T, (to be chosen below) the closed ball

$$B_{T,\epsilon} = \left\{ \rho \in X_T : \left\| \rho \right\|_{X_T} \leq \nu^{-1/4} \epsilon \right\}.$$

The first main lemma is to prove that the first term can be made small by choosing T small. This is actually quite important, and note that we do not get an explicit estimate on the convergence rate.

lem:HeatCont **Lemma 7.3.** *For all $\epsilon > 0$ and $u_0 \in \dot{H}^{1/2}$, there is a $T = T(\epsilon, u_0)$ sufficiently small such that*

$$\left\| e^{\nu t \Delta} u_0 \right\|_{X_T} < \nu^{-1/4} \epsilon.$$

Proof. Define $f := e^{\nu t \Delta} u_0$. By the standard energy estimate on the heat equation we have

$$\left\| f(t) \right\|_{\dot{H}^{1/2}}^2 + \nu \int_0^t \left\| f(\tau) \right\|_{\dot{H}^{3/2}}^2 d\tau \leq \left\| u_0 \right\|_{\dot{H}^{1/2}}^2.$$

(for smooth data this is justified by a formal calculation; for general $\dot{H}^{1/2}$ data one can regularize and pass to the limit). In particular, we have

$$\nu \int_0^t \left\| f(\tau) \right\|_{\dot{H}^{3/2}}^2 d\tau \leq \left\| u_0 \right\|_{\dot{H}^{1/2}}^2 - \left\| f(t) \right\|_{\dot{H}^{1/2}}^2. \tag{7.2}$$ ineq:heattriv

The next claim is that the solution operator to the heat equation is continuous at $t = 0$ in $\dot{H}^{1/2}$:

$$\lim_{t \to 0^+} \left\| f(t) - u_0 \right\|_{\dot{H}^{1/2}} = 0. \tag{7.3}$$ ineq:heatcont

This can be verified from the dominated convergence theorem on the frequency side, noting that:

$$\left\| f(t) - u_0 \right\|_{\dot{H}^{1/2}}^2 = \int |\eta| \left| e^{-\nu t |\eta|^2} - 1 \right|^2 \left| \widehat{u_0}(\eta) \right|^2 d\eta.$$

Also, observe that this convergence is uniform in ν for $\nu \lesssim 1$ (why?). Putting (7.2) and (7.3) together gives us

$$\lim_{t \to 0^+} \nu \int_0^t \left\| f(\tau) \right\|_{\dot{H}^{3/2}}^2 d\tau = 0. \tag{7.4}$$ ineq:L2H32heat

By interpolation,

$$\begin{aligned}\left\|e^{\nu t\Delta}u_0\right\|_{X_T} &\leq \left\|e^{t\Delta}u_0\right\|^{1/2}_{L^\infty(0,T;\dot{H}^{1/2})}\left\|e^{t\Delta}u_0\right\|^{1/2}_{L^2(0,T;\dot{H}^{3/2})}\\ &\leq \nu^{-1/4}\left\|u_0\right\|^{1/2}_{\dot{H}^{1/2}}\left(\sqrt{\nu}\left\|e^{t\Delta}u_0\right\|_{L^2(0,T;\dot{H}^{3/2})}\right)^{1/2},\end{aligned}$$

which, together with (7.4), completes the lemma. □

Next, we control the nonlinear term.

lem:Bctrl **Lemma 7.4.** *For all $f, g \in X_T$ and $T > 0$ we have the following (the implicit constant is independent of T, f, g, ν),*

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{X_T} \lesssim \nu^{-3/4}\left\|g\right\|_{X_T}\left\|f\right\|_{X_T}.$$

Proof. By Minkowski, parabolic regularization (see Proposition H.4), and $\dot{H}^s$ boundedness of the Leray projection we have

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{X_T} \lesssim \frac{1}{\nu^{3/4}}\left\|\int_0^t \frac{1}{(t-\tau)^{3/4}}\left\|f(\tau)\otimes g(\tau)\right\|_{\dot{H}^{1/2}}\right\|_{L^4_t(0,T)}$$

Then, by Proposition G.7, we have

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{X_T} \lesssim \frac{1}{\nu^{3/4}}\left\|\int_0^t \frac{1}{(t-\tau)^{3/4}}\left\|f(\tau)\right\|_{\dot{H}^1}\left\|g(\tau)\right\|_{\dot{H}^1}d\tau\right\|_{L^4_t(0,T)}.$$

We have a convolution in time left over. For this we can use the Hardy-Littlewood-Sobolev inequality (which is actually an end-point Young's inequality for convolutions in disguise):

$$\left\|f * |\cdot|^\alpha\right\|_{L^q(\mathbb{R}^d)} \lesssim_{p,q,d,\alpha} \left\|f\right\|_{L^p(\mathbb{R}^d)}, \tag{7.5}$$ ineq:HLS

if $1 < p < q < \infty$, $0 < \alpha < d$ and

$$\frac{1}{p}+\frac{\alpha}{d} = \frac{1}{q}+1.$$

Indeed, (7.5) implies that (with $q = 4$, $\alpha = 3/4$, $d = 1$ and $p = 2$):

$$\begin{aligned}\left\|\int_0^t e^{\nu(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{X_T} &\lesssim \nu^{-3/4}\left(\int_0^t \left\|f(\tau)\right\|^2_{\dot{H}^1}\left\|g(\tau)\right\|^2_{\dot{H}^1}d\tau\right)^{1/2}\\ &\lesssim \nu^{-3/4}\left\|f\right\|_{X_T}\left\|g\right\|_{X_T},\end{aligned}$$

where the last line followed by Cauchy-Schwarz. □

Next, we claim that $\Phi : B_{T,\epsilon} \to B_{T,\epsilon}$ for T, ϵ chosen sufficiently small. First,

$$\left\|v\right\|_{X_T} \leq \left\|e^{\nu t\Delta}u_0\right\|_{X_T} + \left\|\int_0^t e^{\nu(t-\tau)\Delta}B(\rho,\rho)(\tau)d\tau\right\|_{X_T}. \tag{7.6}$$ ineq:v

It follows from Lemma 7.3 that, for all $\nu > 0$ and for all ϵ, there exists a $T(u_0, \epsilon)$ (independent of ν) such that

$$\|v\|_{X_T} \leq \frac{\epsilon}{2\nu^{1/4}} + \left\| \int_0^t e^{(t-\tau)\Delta} B(\rho,\rho)(\tau) d\tau \right\|_{X_T}.$$

By Lemma 7.4, we further get for some $C > 0$ independent of T

$$\|v\|_{X_T} \leq \frac{\epsilon}{2\nu^{1/4}} + \nu^{-3/4} C \|\rho\|_{X_T}^2.$$

Therefore, for $\epsilon < \frac{\nu}{2C}$, we deduce that Φ indeed maps $B_{T,\epsilon}$ to itself. To deduce Lipschitz continuity, consider $\rho_1, \rho_2 \in X_T$ and compute

$$\|\Phi[\rho_1] - \Phi[\rho_2]\|_{X_T} = \left\| \int_0^t e^{\nu(t-\tau)\Delta} \left(B(\rho_1,\rho_1) - B(\rho_2,\rho_2)\right)(\tau) d\tau \right\|_{X_T}.$$

Since B is bilinear we get the following, applying also Lemma 7.4,

$$\begin{aligned} \|\Phi[\rho_1] - \Phi[\rho_2]\|_{X_T} &= \left\| \int_0^t e^{\nu(t-\tau)\Delta} \left(B(\rho_1,\rho_1-\rho_2) - B(\rho_2,\rho_2-\rho_1)\right)(\tau) d\tau \right\|_{X_T} \\ &\leq C\nu^{-3/4} \left(\|\rho_1\|_{X_T} + \|\rho_2\|_{X_T} \right) \|\rho_1 - \rho_2\|_{X_T}. \end{aligned}$$

Therefore for $\|\rho_i\|_{X_T} < \frac{\nu}{2C}$, we have that Φ is Lipschitz continuous with constant strictly less than one. Hence it follows by the contraction mapping principle, Theorem I.1, we have a unique solution u to (7.1) in $B_{\epsilon,T}$. One could worry that there could still be a different solution in $X_T \setminus B_{\epsilon,T}$. We will see below (Theorem 7.16) that for positive times, mild solutions are classical, and hence we only need to worry about a lack of uniqueness at $t = 0$. Since $X_T = L^4(0,T;\dot{H}^1_x)$ involves a time integral, it follows that any function $v \in X_T$ will satisfy

$$\|v\|_{L^4(0,\tilde{T};\dot{H}^1_x)} < \epsilon$$

for $\tilde{T}$ chosen sufficiently small. Hence $v(t) = u(t)$ for $0 \leq t < \min(T, \tilde{T})$ and it follows that mild solutions are unique in X_T.

It remains to show that $u \in C([0,T];\dot{H}^{1/2}_x) \cap L^2(0,T;\dot{H}^{3/2}_x)$ if $u_0 \in \dot{H}^{1/2}$. This will follow from the observation that Φ maps $B_{\epsilon,T}$ into a ball in $C([0,T];\dot{H}^{1/2}_x) \cap L^2(0,T;\dot{H}^{3/2}_x)$ of radius comparable to the initial data u_0. Notice that for all F, if we denote g such that

$$\partial_t g - \Delta g = F,$$

we get for all $s \geq 0$ and $0 \leq t_1 < t_2$ (easiest via energy estimates):

$$\|g(t_2)\|_{\dot{H}^s}^2 + \|g\|_{L^2(t_1,t_2;\dot{H}^{s+1})}^2 \lesssim_\nu \|g(t_1)\|_{\dot{H}^s}^2 + \|F\|_{L^2(t_1,t_2;\dot{H}^{s-1})}^2.$$

It follows that, if we denote $u = \Phi[v]$,

$$\|u(t)\|_{\dot{H}^{1/2}}^2 + \|u\|_{L^2(0,t;\dot{H}^{3/2})}^2 \lesssim \|u_0\|_{\dot{H}^{1/2}}^2 + \|v \otimes v\|_{L^2(0,t;\dot{H}^{1/2})}^2.$$

By Proposition G.7 we get the following for $v \in B_{\epsilon,T}$ and $0 \leq t_1 < t_1 < T$,

$$\begin{aligned} \|v \otimes v\|_{L^2(t_1,t_2;\dot{H}^{1/2})} &\lesssim_\nu \|v\|_{L^4(t_1,t_2;\dot{H}^1)}^2 \lesssim \epsilon^2 \\ \lim_{|t_2-t_1|\to 0} \|v \otimes v\|_{L^2(t_1,t_2;\dot{H}^{1/2})}^2 &\lesssim_\nu \lim_{|t_2-t_1|\to 0} \|v\|_{L^4(t_1,t_2;\dot{H}^1)}^2 = 0. \end{aligned}$$

Hence, we see that $u \in L^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ and

$$\|u\|^2_{L^\infty(0,T;\dot{H}^{1/2})} + \|u\|^2_{L^2(0,T;\dot{H}^{3/2})} \lesssim_\nu \|u_0\|^2_{\dot{H}^{1/2}} + \epsilon^4,$$

from which the desired regularity on any solution to $\Phi(u) = u$ with $u \in B_{\epsilon,T}$ follows. To see continuity, we just note that for $0 \leq t_1 < t_2 < T$, that from the definition of Φ and Lemma 7.4 (or at least the proof) we have

$$\|u(t_1) - u(t_2)\|_{\dot{H}^{1/2}} \lesssim_\nu \left\| e^{\nu(t_2-t_1)\Delta} u(t_2) - u(t_1) \right\|_{\dot{H}^{1/2}} + \|v\|^2_{L^4(t_1,t_2;\dot{H}^1)}.$$

Continuity of the heat semigroup then implies that Φ is a bounded map from $B_{\epsilon,T}$ to $C([0,T];\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2})$ (and hence these bounds are satisfied by the solution u). This completes the proof of local well-posedness for arbitrary data.

To get global existence for small data, simply note that if u_0 is sufficiently small in $\dot{H}^{1/2}$ (specifically $\|u_0\|_{\dot{H}^{1/2}} < \frac{\nu}{2C}$), then we do not need to pick T small in Lemma 7.3, and in fact the result holds true for $T = \infty$. Lemma 7.4 also holds for $T = \infty$ and so the proof goes through as above. □

Exercise 7.5. Prove that there is a maximal existence time $T^\star \leq \infty$ until which the mild solution can be extended uniquely and that if $u \in C([0,T];\dot{H}^{1/2}) \cap L^4(0,T;\dot{H}^1)$ then $T < T^\star$ (NOTE: it is MUCH harder to prove this with $L^\infty(0,T;\dot{H}^{1/2})$ – can you guess why?)

7.3 Local well-posedness in L^3

There is something very clean about the well-posedness in $\dot{H}^{1/2}$ (both the proof and the statement), but it is not the only critical space. For example, all of the following spaces scale the same way (don't worry if you've never heard of the weirder spaces on the left, they are just larger and larger spaces which scale the same way):

$$\dot{H}^{1/2} \subset L^3 \subset L^{3,\infty} \subset BMO^{-1} \subset B^{-1}_{\infty,\infty}.$$

We can refine the $\dot{H}^{1/2}$ to L^3 (a strictly larger space by Sobolev embedding, Proposition G.5) and then we will discuss some further refinements below (see [KT01] and [BP08] for studies of well/ill-posedness in some of the more unusual spaces that appear above).

For local well-posedness in L^3 we will use an argument which is of a slightly different flavor. We will also drop ν for the remainder of the argument for simplicity – one can always add it back in. Define the norm

$$\|u\|_{X_T} = \sup_{0<t<T} t^{1/4} \|u(t)\|_{L^6}.$$

Note that parabolic regularization, Proposition H.4, implies

$$\left\| e^{t\Delta} u_0 \right\|_{L^6} \lesssim \frac{1}{t^{1/4}} \|u_0\|_{L^3},$$

and hence $\left\| e^{t\Delta} f \right\|_{X_T} \lesssim \|f\|_{L^3}$. It turns out that the set of initial data for which $\left\| e^{t\Delta} u_0 \right\|_{X_T} < \infty$ is larger than L^3, in fact this is a Besov space which contains L^3, see [LR02].

This norm will play the role of $L^4(0,T;\dot{H}^1)$ in the $\dot{H}^{1/2}$ local well-posedness argument. As in Theorem 7.1, in order to get local well-posedness for large data, we will need an analogue of Lemma 7.3:

lem:HeatL3Cont **Lemma 7.6.** *For all $f \in L^3$ there holds*

$$\lim_{t \to 0^+} t^{1/4} \left\| e^{t\Delta} f \right\|_{L^6} = 0.$$

Proof. The proof here is based on a similar lemma, [Lemma 4.4, [GMO88]]. Let $\epsilon > 0$ be arbitrary. First, let $N > 0$ be such that $\int_{|x|>N} |f|^3 \, dx < \epsilon^3$ and denote $f \mathbf{1}_{|x| \leq N} = f_1$. By Proposition H.4, we have

$$\liminf_{t \to 0^+} t^{1/4} \left\| e^{t\Delta}(f - f_1) \right\|_{L^6} \lesssim \epsilon.$$

Consider next

$$\begin{aligned} \left(t^{1/4} \left\| e^{t\Delta} f_1 \right\|_{L^6} \right)^6 &= t^{3/2} \left(\int_{|x|>2N} + \int_{|x| \leq 2N} \right) \left(\frac{1}{(4\pi t)^{3/2}} \int_{|y| \leq N} e^{-\frac{|x-y|^2}{4t}} f(y) dy \right)^6 dx \\ &= T_F + T_C. \end{aligned}$$

On the support of the integrand of T_F we have $|x - y| > |x|/2 \geq N$ and hence

$$\begin{aligned} T_F &\lesssim e^{-\frac{6N^2}{8t}} t^{3/2} \int_{|x|>2N} \left(\frac{1}{(4\pi t)^{3/2}} \int_{|y| \leq N} e^{-\frac{|x-y|^2}{8t}} \mathbf{1}_{|x-y|>N} f(y) dy \right)^6 dx \\ &\lesssim e^{-\frac{6N^2}{8t}} \|f\|_{L^3}^6, \end{aligned}$$

which satisfies

$$\lim_{t \to 0^+} T_F = 0.$$

Turn next to T_C. Then

$$\begin{aligned} T_C &\lesssim t^{3/2} \int_{|x| \leq 2N} \left(\frac{1}{(4\pi t)^{3/2}} \int_{|y| \leq N} e^{-\frac{|x-y|^2}{4t}} \mathbf{1}_{|x-y| \leq \delta} f(y) dy \right)^6 dx \\ &\quad + t^{3/2} \int_{|x| \leq 2N} \left(\frac{1}{(4\pi t)^{3/2}} \int_{|y| \leq N} e^{-\frac{|x-y|^2}{4t}} \mathbf{1}_{|x-y| > \delta} f(y) dy \right)^6 dx \\ &= T_{CC} + T_{CF}. \end{aligned}$$

The T_{CF} term is treated the same as T_F above. For T_{CC}, by Hölder's inequality, we have

$$\begin{aligned} T_{CC} &\lesssim t^{3/2} \int_{|x| \leq 2N} \left(\frac{1}{(4\pi t)^{3/2}} \int_{|y| \leq N} e^{-\frac{|x-y|^2}{4t}} \mathbf{1}_{|x-y| \leq \delta} f(y) dy \right)^6 dx \\ &\lesssim t^{3/2} \int_{|x| \leq 2N} \left(\int_{|x-y| \leq \delta} |f(y)|^3 \, dy \right) \left(\int_{|y| \leq N} \left(\frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x-y|^2}{4t}} \mathbf{1}_{|x-y| \leq \delta} \sqrt{|f(y)|} \right)^{6/5} dy \right)^5 dx \\ &\lesssim \sup_{x : |x| \leq 2N} \left(\int_{|x-y| \leq \delta} |f(y)|^3 \, dy \right) \left\| e^{\frac{5}{6} t\Delta} |f|^{3/5} \right\|_{L^5}^5 \\ &\lesssim \sup_{x : |x| \leq 2N} \left(\int_{|x-y| \leq \delta} |f(y)|^3 \, dy \right) \|f\|_{L^3}^3 . \end{aligned}$$

It suffices to prove that

$$\lim_{\delta\to 0^+}\sup_{x:|x|\le 2N}\left(\int_{|x-y|\le\delta}|f(y)|^3\,dy\right)=0.$$

Suppose this were incorrect. Then there would exist an $\epsilon_0>0$ and sequences $\{x_n\}_{n=1}^\infty\subset\{x:|x|\le 2N\}$, $\{\delta_n\}_{n=1}^\infty$ with $\delta_n\to 0$ such that

$$\int_{|x_n-y|\le\delta_n}|f(y)|^3\,dy>\epsilon_0.$$

By compactness, there would exist a subsequence $\{x_{n_k}\}_{k=1}^\infty$ and $x_\infty\in B(0,2N)$ with $\lim_{k\to\infty}x_{n_k}=x_\infty$. Note that for any $\delta>0$, $B(x_{n_k},\delta_{n_k})\subset B(x_\infty,\delta)$ for all k sufficiently large. However, $\epsilon_0>0$ contradicts the fact that $|f|^3\in L^1$, since this implies

$$\lim_{\delta\to 0}\int_{|x_\infty-y|\le\delta}|f(y)|^3\,dy=0.$$

This completes the proof. □

It will be of interest also to quantify the instant regularization of the heat equation in a different scale invariant way. We can use $L^5_tL^5_x$, which will provide another kind of norm to measure.

lem:L55cont **Lemma 7.7.** *Let $u_0\in L^3$. Then, there is a constant C, independent of T, such that*

$$\left\|e^{t\Delta}u_0\right\|_{L^5(0,T;L^5)}\le C\left\|u_0\right\|_{L^3}.$$

In particular, for all $\epsilon>0$, there exists a time $T(u_0,\epsilon)>0$ such that

$$\left\|e^{t\Delta}u_0\right\|_{L^5(0,T;L^5)}<\epsilon.$$

Proof. Denote $u(t)=e^{t\Delta}u_0$. Without loss of generality, we may assume that $u_0\ge 0$ (as the PDE is linear, we can evolve the positive and negative parts separately) Begin with an L^3 energy estimate on the heat equation:

$$\begin{aligned}\frac{1}{3}\frac{d}{dt}\|u(t)\|_{L^3}^3&=\int u^2\Delta u dx\\&=-2\int u\,|\nabla u|^2\\&=-\frac{8}{9}\int\left|\nabla u^{3/2}\right|^2dx.\end{aligned}$$

Now we use yet another kind of Gagliardo-Nirenberg-Sobolev inequality (see the HW exercises)

$$\left\|u^{3/2}\right\|_{L^{10/3}}\lesssim\left\|u^{3/2}\right\|_{L^2}^{2/5}\left\|\nabla u^{3/2}\right\|_{L^2}^{3/5}.$$

However, after taking squares, this is the same as

$$\|u\|_{L^5}^5\lesssim\|u\|_{L^3}^2\left\|\nabla u^{3/2}\right\|_{L^2}^2.$$

Using that $\|u(t)\|_{L^3} \leq \|u_0\|_{L^3}$ we get, for solutions of the heat equation,

$$\|u(t)\|_{L^5}^5 \lesssim \|u_0\|_{L^3}^2 \left\|\nabla u^{3/2}(t)\right\|_{L^2}^2 .$$

Therefore, we get

$$\frac{1}{3}\frac{d}{dt}\|u(t)\|_{L^3}^3 \lesssim -\frac{1}{\|u_0\|_{L^3}^2}\|u(t)\|_{L^5}^5 .$$

Integrating, gives

$$\|u(t)\|_{L^3}^3 + \frac{1}{\|u_0\|_{L^3}^2}\|u(t)\|_{L^5(0,T;L^5)}^5 \lesssim \|u_0\|_{L^3}^3 ,$$

and so the result follows. □

From the previous lemma, we can prove local well-posedness in L^3 using a proof similar to that of Theorem 7.1. This was first proved by Kato in 1984 [Kat84].

thm:FKHL3 **Theorem 7.8** (Kato [Kat84])**.** *Let $u_0 \in L^3_\sigma$. Then, there exists a $0 < T = T(u_0)$ and a unique mild solution to (7.1) in the class $C([0,T];L^3) \cap X_T \cap L^5(0,T;L^5_x)$. Moreover, the solution map $M : L^3 \to X_T$ is locally Lipschitz continuous with respect to the initial data. Finally, there exists a universal $\epsilon_0 > 0$ such that if $\|u_0\|_{L^3} < \epsilon_0$, then the mild solution is global in time.*

Proof. For $u_0 \in L^3_\sigma$ and $\rho \in X_T \cap L^5(0,T;L^5)$, define the mapping $\Phi[\rho] \to v$ given by

$$v(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}B(\rho,\rho)(\tau)d\tau. \qquad (7.7)$$ eq:L3mild

Define for ϵ, T, (to be chosen below) the closed ball

$$B_{T,\epsilon} = \left\{\rho \in X_T : \|\rho\|_{X_T} \leq \epsilon, \|\rho\|_{L^5(0,T;L^5)} \leq \epsilon\right\}.$$

equipped with the topology induced by the norm

$$\|\rho\|_* = \|\rho\|_{X_T} + \|\rho\|_{L^5(0,T;L^5)} .$$

Actually, we can do the entire argument without the $L^{5,5}_{t,x}$ control, however, we want to show both arguments at once.

The initial data term in (7.7) will be controlled by Lemmas 7.6 and 7.7. The nonlinear term in X_T is controlled via

lem:BctrlL3 **Lemma 7.9.** *For all $f, g \in X_T$ and $T > 0$ we have the following (the implicit constant is independent of T, f, g),*

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{X_T} \lesssim \|g\|_{X_T}\|f\|_{X_T} .$$

Proof. By Minkowski's inequality, Proposition H.4, and the L^p boundedness of the Leray projection, which comes from Theorem F.4,

$$\begin{aligned} t^{1/4}\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^6} &\lesssim t^{1/4}\int_0^t \left\|e^{(t-\tau)\Delta}B(f,g)(\tau)\right\|_{L^6} d\tau \\ &\lesssim t^{1/4}\int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}}\left\|\mathbb{P}(f\otimes g)(\tau)\right\|_{L^3} d\tau \\ &\lesssim t^{1/4}\int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}}\left\|(f\otimes g)(\tau)\right\|_{L^3} d\tau \\ &\lesssim t^{1/4}\int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}}\left\|f(\tau)\right\|_{L^6}\left\|g(\tau)\right\|_{L^6} d\tau \\ &\lesssim \|f\|_{X_T}\|g\|_{X_T}\int_0^t \frac{t^{1/4}}{(t-\tau)^{\frac{3}{4}}\tau^{1/2}}d\tau. \end{aligned}$$

Note that

$$\int_0^t \frac{t^{1/4}\tau^{1/2}}{(t-\tau)^{\frac{3}{4}}}d\tau = \int_0^{t/2}\frac{t^{1/4}}{(t-\tau)^{\frac{3}{4}}\tau^{1/2}}d\tau + \int_{t/2}^t \frac{t^{1/4}}{(t-\tau)^{\frac{3}{4}}\tau^{1/2}}d\tau \lesssim 1.$$

This completes the proof of Lemma 7.9. □

The $L^5(0,T;L^5)$ norm of the nonlinear term is controlled via

lem:BctrlL55 **Lemma 7.10.** *For all $f,g\in L^5(0,T;L^5)$ and $T>0$ we have the following (the implicit constant is independent of T,f,g),*

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^5_t(0,T;L^5_x)} \lesssim \|g\|_{L^5(0,T;L^5)}\|f\|_{L^5(0,T;L^5)}.$$

Proof. From Minkowksi's inequality and parabolic regularization,

$$\begin{aligned} \left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^5_t(0,T;L^5_x)} &\lesssim \left\|\int_0^t \left\|e^{(t-\tau)\Delta}B(f,g)(\tau)\right\|_{L^5_x} d\tau\right\|_{L^5_t(0,T)} \\ &\lesssim \left\|\int_0^t \frac{1}{(t-\tau)^{1/2+\frac{3}{10}}}\left\|\mathbb{P}(f\otimes g)(\tau)\right\|_{L^{5/2}_x} d\tau\right\|_{L^5_t(0,T)}. \end{aligned}$$

By the $L^{5/2}$ boundedness of the Leray projection and Cauchy-Schwarz,

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^5_t(0,T;L^5_x)} \lesssim \left\|\int_0^t \frac{1}{(t-\tau)^{1/2+\frac{3}{10}}}\|f(\tau)\|_{L^5_x}\|g(\tau)\|_{L^5_x} d\tau\right\|_{L^5_t(0,T)}.$$

Now, we use the Hardy-Littlewood-Sobolev inequality in time again as above in (7.5) and we deduce

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^5_t(0,T;L^5_x)} \lesssim \left\|\|f(\tau)\|_{L^5_x}\|g(\tau)\|_{L^5_x}\right\|_{L^{5/2}_t(0,T)},$$

which by Cauchy-Schwarz is the desired estimate

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(f,g)(\tau)d\tau\right\|_{L^5_t(0,T;L^5_x)} \lesssim \|f\|_{L^5(0,T;L^5)}\|g\|_{L^5(0,T;L^5)},$$

□

The first step is to prove that $\Phi : B_{\epsilon,T} \to B_{\epsilon,T}$. By Lemma 7.6, for all ϵ, there exists a T such that

$$\left\|e^{t\Delta}u_0\right\|_{X_T} < \frac{\epsilon}{2}.$$

Similarly, by Lemma 7.7, for all ϵ, there exists a T such that we also have

$$\left\|e^{t\Delta}u_0\right\|_{L^5(0,T;L^5)} < \frac{\epsilon}{2}.$$

By Lemma 7.9, there exists a $C > 0$ such that

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(\rho,\rho)(\tau)d\tau\right\|_{X_T} \le C\left\|\rho\right\|_{X_T}^2.$$

and by Lemma 7.10 there exists a $C > 0$ (we may as well take the same C; e.g. the max of the two inequalities)

$$\left\|\int_0^t e^{(t-\tau)\Delta}B(\rho,\rho)(\tau)d\tau\right\|_{L^5_t(0,T;L^5)} \le C\left\|\rho\right\|_{L^5(0,T;L^5)}^2.$$

Hence, for $\epsilon < \frac{1}{2C}$, $\Phi : B_{\epsilon,T} \to B_{\epsilon,T}$. Similarly, for $\epsilon < \frac{1}{2C}$ we have that Φ is a contraction on $B_{\epsilon,T}$ because

$$\begin{aligned}
\left\|\Phi[\rho_1] - \Phi[\rho_2]\right\|_{X_T} &\le C\left(\left\|\rho_1\right\|_{X_T} + \left\|\rho_2\right\|_{X_T}\right)\left\|\rho_1 - \rho_2\right\|_{X_T} \\
\left\|\Phi[\rho_1] - \Phi[\rho_2]\right\|_{L^5(0,T;L^5)} &\le C\left(\left\|\rho_1\right\|_{L^5(0,T;L^5)} + \left\|\rho_2\right\|_{L^5(0,T;L^5)}\right)\left\|\rho_1 - \rho_2\right\|_{L^5(0,T;L^5)}.
\end{aligned}$$

By the contraction mapping principle, Theorem I.1, we have a unique mild solution in $B_{\epsilon,T}$, which we denote $u(t)$. Furthermore, as in the proof of Theorem I.2, the solution map $M : L^3_\sigma \to X_T$ is locally Lipschitz continuous. Moreover, we note that for all $\delta > 0$ we can choose a T_δ sufficiently small such that

$$\left\|u\right\|_{X_{T_\delta}} + \left\|u\right\|_{L^5(0,T_\delta;L^5)} < \delta. \tag{7.8}$$

ineq:L3small

Next, we verify that $u \in L^\infty_t L^3_x$, by proving that $\Phi : B_{\epsilon,T} \to L^\infty_t L^3_x$. Of course we have

$$\left\|e^{t\Delta}u_0\right\|_{L^3} \lesssim \left\|u_0\right\|_{L^3}.$$

Then, again using the L^p boundedness of the Leray projection from Theorem F.4,

$$\begin{aligned}
\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau\right\|_{L^3} &\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2}}\left\|(u\otimes u)(\tau)\right\|_{L^3} d\tau \\
&\lesssim \int_0^t \frac{1}{(t-\tau)^{1/2}}\left\|u(\tau)\right\|_{L^6}^2 d\tau \\
&\lesssim \left\|u\right\|_{X_T}^2 \int_0^t \frac{1}{(t-\tau)^{1/2}\tau^{1/2}}d\tau \\
&\lesssim \left\|u\right\|_{X_T}^2.
\end{aligned}$$

We next want to improve $L^\infty(0,T;L^3)$ to $C([0,T];L^3)$. We will see below that mild solutions are $C^\infty_{t,x}$ for $t > 0$, so the only issue with continuity in L^3 is at $t = 0$ (exercise: verify that this argument is not circular). As above,

$$\sup_{0<t<T}\left\|u(t) - e^{t\Delta}u_0\right\|_{L^3} \lesssim \left\|u\right\|_{X_T}^2.$$

However, by the observation in (7.8) above, this implies $\lim_{t\to 0^+} \left\| u(t) - e^{t\Delta}u_0 \right\|_{L^3} = 0$ which in turn implies continuity at zero.

Next we argue that the solution is unique in $C([0,T];L^3)\cap X_T$, for which it is sufficient to verify (7.8). In order to do so, we can deduce some additional integrability. Suppose $u \in C([0,T];L^3)\cap X_T$ solves (7.1). Let $6 < p < \infty$ be fixed arbitrary and denote

$$p' = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right).$$

Then by Proposition H.4 (and the L^3 boundedness of the Leray projections),

$$\begin{aligned} t^{p'} \left\| u(t) \right\|_{L^p} &\lesssim \left\| u_0 \right\|_{L^3} + t^{p'} \int_0^t \frac{1}{(t-\tau)^{1/2+p'}} \left\| u(\tau) \right\|_{L^6}^2 d\tau \\ &\lesssim \left\| u_0 \right\|_{L^3} + \left\| u \right\|_{X_T}^2 t^{p'} \int_0^t \frac{1}{(t-\tau)^{1/2+p'}\tau^{1/2}} d\tau \\ &< \infty. \end{aligned}$$

By L^3 continuity of u at zero we have

$$\lim_{t\to 0^+} \left\| u(t) - e^{t\Delta}u_0 \right\|_{L^3} = 0.$$

By interpolation, for $\theta^{-1} = 6\left(\frac{1}{3} - \frac{1}{p}\right) > 1$,

$$\begin{aligned} t^{1/4} \left\| u(t) - e^{t\Delta}u_0 \right\|_{L^6} &\leq \left(t^{\frac{1}{4\theta}} \left\| u - e^{t\Delta}u_0 \right\|_{L^p} \right)^{\theta} \left\| u - e^{t\Delta}u_0 \right\|_{L^3}^{1-\theta} \\ &\lesssim \left\| u - e^{t\Delta}u_0 \right\|_{L^3}^{1-\theta}, \end{aligned}$$

where in the second inequality we used $\frac{1}{4\theta} = p'$. Therefore, for any mild solution in $u \in C([0,T];L^3)\cap X_T$,

$$\lim_{t\to 0^+} t^{1/4} \left\| u(t) - e^{t\Delta}u_0 \right\|_{L^6} = 0.$$

By Lemma 7.6, it follows that

$$\lim_{t\to 0} t^{1/4} \left\| u(t) \right\|_{L^6} = 0,$$

and so every mild solution in $u \in C([0,T];L^3)\cap X_T$ satisfies (7.8), which is sufficient for uniqueness by the contraction mapping theorem (in particular, this shows that all mild solutions are in the same ball for T small enough and hence must coincide).

The global existence for small data follows as in Theorem 7.1 above. This completes the proof of Theorem 7.8. □

Remark 7.11. Note that the proof shows the improved result on the instant gain of L^p integrability: for all $3 < p < \infty$,

$$\lim_{t\to 0} t^{\frac{3}{2}\left(\frac{1}{3}-\frac{1}{p}\right)} \left\| u(t) \right\|_{L^p} = 0.$$

7.4 Continuation and regularity

There are two questions left: (A) we have yet to see why mild solutions are instantly $C^\infty_{t,x}$ (although we have seen a hint of this above) and (B) we do not know how far solutions can be extended. The existence times in Theorems 7.1 and 7.8 depend on the *rate* at which $\left\|e^{t\Delta}u_0\right\|_{L^4(0,T;\dot{H}^1)}$ or $t^{1/4}\left\|e^{t\Delta}u_0\right\|_{L^6}$ go to zero as $t \to 0^+$. As such, it is not a priori clear that knowing $u \in L^\infty(0,T;\dot{H}^{1/2})$ or $u \in L^\infty(0,T;L^3)$ is enough to propagate mild solutions for $t > T$. This is indeed true, but it is hard to prove (this was first proved in [ESS03]). However, we do have the lemma

Lemma 7.12. *Let $u_0 \in L^3_\sigma$ and let u be the unique, local-in-time, mild solution in $C([0,T];L^3) \cap X_T$ associated with this initial data, which exists at least for some time $T > 0$. There exists a $0 < T^\star \leq \infty$ such that the solution can be uniquely continued as a mild solution in $C([0,T];L^3) \cap X_T$ for all $T < T^\star$. Moreover, if $T^\star < \infty$, then*

$$u \notin C([0,T^\star];L^3).$$

That is, the solution can be continued past $T^\star$ if it is continuous up to $T^\star$.

Proof. Suppose that there is some $f \in L^3$ such that $u(t) \to f$ as $t \nearrow T$. The local existence time of a solution for a given initial data depends only on the relationship between ϵ and T in Lemma 7.6. Write for any $\tau > 0$ and t close to T:

$$\begin{aligned}\tau^{1/4}\left\|e^{\tau\Delta}u(t)\right\|_{L^6} &\leq \tau^{1/4}\left\|e^{\tau\Delta}f\right\|_{L^6} + \tau^{1/4}\left\|e^{\tau\Delta}\left(u(t)-f\right)\right\|_{L^6} \\ &\lesssim \tau^{1/4}\left\|e^{\tau\Delta}f\right\|_{L^6} + \left\|u(t)-f\right\|_{L^3}.\end{aligned}$$

Hence, given an $\epsilon > 0$ we can pick t such that $\left\|u(t)-f\right\|_{L^3} < \epsilon/2$ is small and then pick τ sufficiently small depending only on f to make the first term less than $\epsilon/2$. Hence, the existence time from the Kato fixed point argument will be uniform as $t \nearrow T$ and so we can extend the solution for some times past T. Therefore, we can define the maximal existence time as

$$T^\star = \sup\left\{T : u \in C([0,T];L^3)\right\}.$$

□

The next lemma is a bit subtle, it shows that for a subcritical L^q spaces, $q > 3$, we can get a simple quantitative estimate on the existence time based on the norm of the initial data, something which we know is not possible in the critical space L^3.

em:UniformSubExistence **Lemma 7.13.** *Let $u_0 \in L^q \cap L^3_\sigma$. For all $M > 0$, there exists a $T = T(M) > 0$ such that the unique mild solution u with maximal existence time $T^\star$ satisfies $T(M) < T^\star$ if $\left\|u_0\right\|_{L^q} \leq M$. That is, the solution exists at least as long as $T(M)$.*

Remark 7.14. Note that the estimated existence time does not depend on the L^3 norm of the initial data.

Proof. Recall that the estimated time of existence in the contraction mapping scheme depended only on how small one has to pick T such that $\left\|e^{t\Delta}u\right\|_{X_T} < \epsilon_0$ for some fixed ϵ_0 depending only on the constants in some functional inequalities. In our work, we chose $\left\|u\right\|_{X_T} = \sup_{0<t<T} t^{1/4}\left\|u(t)\right\|_{L^6}$. If $q \leq 6$ then by Proposition H.4,

$$t^{1/4}\left\|e^{t\Delta}u_0\right\|_{L^6} \lesssim t^{\frac{1}{4}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{6}\right)}\left\|u_0\right\|_{L^q}.$$

Then note that if $3 < q \leq 6$, the exponent on the RHS is still positive. Hence, in order to prove that the solution exists up to time T, we just need to pick time such that

$$T^{\frac{1}{4}-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{6}\right)} < \epsilon_0 \left\| u_0 \right\|_{L^q}^{-1}.$$

This proves the lemma for $q \in (3, 6]$. In the case of other q, we just note that the choice of L^6 was arbitrary in the proof of local existence; any $3 < q < \infty$ will suffice (with a bit of computation). Therefore, the lemma will follow similarly in all such cases. □

The last lemma immediately implies the following:

Lemma 7.15. *Let $u_0 \in L^3_\sigma$ and let u be the unique, local-in-time, mild solution in $C([0,T]; L^3) \cap X_T$ associated with this initial data, which exists at least for some time $T > 0$. There exists a $0 < T^\star \leq 0$ such that the solution can be uniquely continued as a mild solution on $C([0,T]; L^3) \cap X_T$ for all $T < T^\star$. Moreover, if $T^\star < \infty$ then for all $q \in (3, \infty]$,*

$$\lim_{t \nearrow T^\star} \left\| u(t) \right\|_{L^q} = +\infty.$$

That is, the solution can be continued past $T^\star$ if it is bounded in any L^q space for $q > 3$ up to $T^\star$.

Proof. Exercise! □

We can make the above more precise. The continuation criterion (7.9) below is also a sufficient criterion for deducing that suitably defined weak solutions are regular, and can also be localized, however, we will present them here as a continuation criterion for mild solutions, which is (much) simpler. Proofs of various results related to Theorem 7.16 are due to Prodi [Pro59b], Serrin, [Ser62b], Ladyzhenskaya [Lad67b], and Escauriaza, Seregin, and Sverak [ESS03].

thm:ProdiSerrinMild **Theorem 7.16** (Prodi-Serrin continuation criteria)**.** *Let $u_0 \in L^3_\sigma$ and let u be the unique, local-in-time, mild solution in $C([0,T]; L^3) \cap X_T$ associated with this initial data with maximal existence time $T^\star$. If $T^\star < \infty$, then for p, q satisfying $2 \leq p \leq \infty$, $3 < q \leq \infty$, and,*

$$\frac{2}{p} + \frac{3}{q} \leq 1, \tag{7.9}$$ cond:ProdiSerrin

there holds

$$\left\| u \right\|_{L^p(0,T^\star;L^q)} = +\infty.$$

Remark 7.17. This theorem is also true for $q = 3$, but this is a much deeper result, proved relatively recently in [ESS03].

Remark 7.18. Note that all the $L^p L^q$ spaces which acheive equality in (7.9) are invariant under the Navier-Stokes scaling. That is, if $u_\lambda(t,x) = \lambda^{-1}(t\lambda^{-2}, x\lambda^{-1})$ then

$$\left\| u_\lambda \right\|_{L^p(0,T\lambda^2;L^q)} = \left\| u \right\|_{L^p(0,T;L^q)}.$$

Exercise 7.19. Find a function $u \in L^\infty(0,1; L^3)$ such that $u \notin C([0,1); L^3)$. *Hint: don't get fancy.*

Remark 7.20. Together with the regularity theorem below, we see that in order to prove global regularity for the 3D Navier-Stokes equations, it suffices to control any norm $\left\| u \right\|_{L^p(0,T;L^q)}$ which satisfies (7.9). These conditions are called the Prodi-Serrin conditions, and, of course, they are supercritical with respect to the $L^\infty_t L^2_x \cap L^2_t(0,T; \dot{H}^1_x)$ norm and all known conserved or dissipated quantities; there are only a few scenarios in which we can prove such norms are finite for all T. Notice that in terms of spatial regularity, these norms are much weaker than the BKM criteria for the Euler equations in §4.

Proof. If $q > 3$ then the argument can be made from from a very slick (but still very illuminating) scaling argument. The endpoint $(p, q) = (\infty, 3)$ is MUCH harder, and was only resolved relatively recently [ESS03]. We will only show the proof of the $q > 3$ criterion.

Suppose that $u(t, x)$ is a mild solution which blows up at some time $T^\star < \infty$. Let $\{t_n\}_{n\geq 1}$ be an arbitrary sequence of times such that $t_n \nearrow T^\star$. We know now that, for $q > 3$,

$$\lim_{n\to\infty} \|u(t_n)\|_{L^q} = \infty,$$

as otherwise we could extend the solution past $T^\star$ (using $q > 3$). Now, define the renormalized set of solutions to Navier-Stokes on the interval $[0, (T^\star - t_n)\lambda_n^2)$ given by

$$u_n(t, x) = \frac{1}{\lambda_n} u\left(\frac{t}{\lambda_n^2} + t_n, \frac{x}{\lambda_n}\right).$$

where λ_n is defined such that

$$\|u_n(0)\|_{L^q} = 1.$$

This implies

$$\lambda_n = \|u(t_n)\|_{L^q}^{\frac{1}{1-3/q}}.$$

By Lemma 7.13, since $q > 3$, there is a $T(1)$ such that if $\|u_n(0)\|_{L^q} \leq T(1)$ then there exists a unique mild solution for times $t < T(1)$. Hence, we know that $u_n(t)$ must exist for all $0 < t < T(1)$, uniformly in n. This implies that

$$(T^\star - t_n)\lambda_n^2 \gtrsim 1,$$

which, by the definition of λ_n, translates to

$$\|u(t_n)\|_{L^q} \gtrsim \frac{1}{(T^\star - t_n)^{\frac{1-3/q}{2}}}.$$

Since this is true for every sequence of t_n, we must have

$$\|u(t)\|_{L^q} \gtrsim (T^\star - t)^{-\frac{1-3/q}{2}}.$$

This is interesting in its own right: it shows that *if* blow-up occurs, it must occur at a speed which is *at least* that specific rate – it could be much faster but cannot be slower (there are examples of PDEs where we know of blow-ups that are much faster than what this scaling trick suggests). However, the right hand side is *not* L^p integrable in time precisely when

$$\frac{p}{2}\left(1 - \frac{3}{q}\right) \geq 1,$$

which is exactly to say

$$1 \geq \frac{3}{q} + \frac{2}{p},$$

which is the critical Prodi-Serrin condition precisely. □

Finally, let us sketch the proof of $C^\infty_{t,x}$ regularity for mild solutions.

Theorem 7.21. *Let $u_0 \in L^3_\sigma \cap L^2$ and let u be the unique, local-in-time, mild solution in $C([0,T];L^3)\cap X_T$ associated with this initial data with maximal existence time $T^\star$. Then $u \in C^\infty_{t,x}((0,T^\star)\times\mathbb{R}^3)$ and hence is a classical solution for $0 < t < T^\star$.*

Remark 7.22. We see that mild solutions are very similar to strong or classical solutions and can be considered a natural extension of strong solutions to rougher data while still enjoying nearly all of the nice properties of strong solutions.

Proof. This is proved by an iteration scheme.

Exercise 7.23. Prove that for initial data $u_0 \in L^3_\sigma \cap L^2$, the mild solution satisfies $u_0 \in L^\infty(0,T;L^2)$ (as it should).

We have already seen that for all $\epsilon > 0$ $u \in L^\infty(\epsilon, T^\star - \epsilon; L^q)$ for all $q \geq 2$. By uniqueness we can write

$$u(t) = e^{(t-\epsilon)\Delta}u(\epsilon) - \int_\epsilon^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau.$$

For $t \geq 2\epsilon$ we then have,

$$\begin{aligned}\|u(t)\|_{L^\infty} &\lesssim \epsilon^{-1/8}\|u\|_{L^\infty(\epsilon,t;L^{12})} + \int_\epsilon^t \frac{1}{(t-\tau)^{\frac{1}{2}+\frac{1}{4}}}\|(u\otimes u)(\tau)\|_{L^6}^2\,d\tau \\ &\lesssim \epsilon^{-1/8}\|u\|_{L^\infty(\epsilon,t;L^{12})} + \|u\|^2_{L^\infty(\epsilon,t;L^{12})}.\end{aligned}$$

Hence, we see that any mild solution will be $L^\infty(\epsilon, T^\star-\epsilon; L^\infty)$ for all $0<\epsilon<T^\star$. Now let us enter the Sobolev scale; we proceed by induction. Consider $H^{1/2}$ first. Then, by Proposition H.4, (for $t \geq 2\epsilon$),

$$\begin{aligned}\|u(t)\|_{H^{1/2}} &\lesssim \epsilon^{-1/2}\|u(\epsilon)\|_{L^2} + \int_\epsilon^t \frac{1}{(t-\tau)^{3/4}}\|u\otimes u\|_{L^2}\,d\tau \\ &\lesssim \epsilon^{-1/2}\|u(\epsilon)\|_{L^2} + \|u\|^2_{L^\infty(\epsilon,t;L^4)}.\end{aligned}$$

Now suppose that $u \in L^\infty(\epsilon, T^\star-\epsilon; H^{k/2})$ for $k \geq 1$. Then, by the Sobolev product rule, Proposition G.8, we get

$$\begin{aligned}\|u(t)\|_{H^{(k+1)/2}} &\lesssim \epsilon^{-1/2}\|u(\epsilon)\|_{H^{k/2}} + \int_\epsilon^t \frac{1}{(t-\tau)^{3/4}}\|(u\otimes u)(\tau)\|_{H^{k/2}}\,d\tau \\ &\lesssim \epsilon^{-1/2}\|u(\epsilon)\|_{H^{k/2}} + \int_\epsilon^t \frac{1}{(t-\tau)^{3/4}}\|u(\tau)\|_{L^\infty}\|u(\tau)\|_{H^{k/2}}\,d\tau \\ &\lesssim \epsilon^{-1/2}\|u(\epsilon)\|_{H^{k/2}} + \|u\|_{L^\infty(\epsilon,t;L^\infty)}\|u\|_{L^\infty(\epsilon,t;H^{k/2})}.\end{aligned}$$

Hence, by induction, $u \in L^\infty(\epsilon, T^\star-\epsilon; H^{k/2})$ for all $k \geq 1$ and all $\epsilon > 0$. To get infinite differentiability in time, we use the PDE and the deduced spatial regularity. □

Exercise 7.24. Prove that if $u_0 \in L^2 \cap L^3_\sigma$ then the resulting unique mild solution is also a Leray-Hopf weak solution.

7.5 Even lower critical regularity

One can use variations of the L^3 proof to expand the class of initial data even further past L^3. For example, consider the larger space define by the class of initial data to the heat equation that will give *finite* X_T norm

$$X_T := \left\{ f : \|f\|_{X_T} = \sup_{0<t<T} t^{1/4} \left\|e^{t\Delta} f\right\|_6 < \infty \right\}.$$

By Lemma 7.6, $L^3 \subset X_T$, however, there exists functions in X_T which are not in L^3, for example, $f(x) = |x|^{-1}$. In the space X_T, we cannot hope to use the proof of Theorem 7.8 to get local well-posedness for large data, because choosing T small does not make the norm of f any smaller in general (Lemma 7.6 shows that this is true if $f \in L^3$ but not in this larger space). However, for small data we can still get results.

Exercise 7.25. Prove that there exists an $\epsilon_0 > 0$ such that if $\sup_{t\in(0,\infty)} t^{1/4} \left\|e^{t\Delta} u_0\right\|_{L^6} \leq \epsilon_0$, then there exists a global-in-time mild solution to the 3D NSE with initial data u_0. In what sense does the solution obtain the initial data? In what class can you prove uniqueness?

As we can see, in larger spaces than L^3, the proof we used in Theorem 7.8 can break down – because the analogue of Lemma 7.3 is no longer true and there is no norm we can use to close a contraction such that picking T smaller makes the norm smaller! In these larger critical spaces, we derive a result only for *small data* (and, as above, the only solutions we construct are global in time). There has been some recent work towards proving that the equations are ill-posed for larger data in $L^{3,\infty}$ [JS15], but this is still currently open. In the space $B^{-1}_{\infty,\infty}$ the situation is worse: the 3D Navier-Stokes equations is ill-posed even for small data [BP08]. Not all critical spaces are equal – this is not Navier-Stokes specific, but a general property of local well-posedness at critical regularity.

Another fun exercise is considering the 2D Navier-Stokes equations in vorticity form. In this case, the critical regularity is L^1. In this case, one can build a local well-posedness theory for initial vorticity in L^1 and even the space of finite Borel measures; see [GG05] and the references therein.