MSRI Summer School: Incompressible Fluid Flows at High Reynolds Number
Howmework 2
July 27, 2015

All references to exercise, chapter, or appendix numbers to the *lecture notes* made available by the instructors.

1. Navier-Stokes

EXERCISE 1.1 (**). *Complete the proof of d'Alambert's paradox, that is do Exercises 2.2 and 2.3 in Section 2 of the lecture notes.*

EXERCISE 1.2 (*). *Prove the exponential decay of energy in 2D unforced NSE on the torus under the mean free condition, i.e. do Exercise 2.5 in Section 2 of the lecture notes. What happens as $t \to \infty$ if we add a time independent $f \in L^2(\mathbb{T}^2)$ on the RHS of the equations?*

EXERCISE 1.3 (*). *Prove the decay of enstrophy in 2D unforced NSE on the plane, i.e. do Exercise 2.5 in Section 2 of the lecture notes.*

EXERCISE 1.4 (***). *Let $u_0 \in L^2_\sigma(\mathbb{T}^2)$. Use Galerkin to prove the global existence of weak solutions that are strong for $t > 0$. That is, prove Theorem 6.8 in Section 6 of the lecture notes.*

EXERCISE 1.5 (*). *Let $f \in C_c^\infty(\mathbb{R}^3)$ and consider for $t \in [0,1)$*

$$u(t,x) = \frac{1}{1-t} f\left(\frac{t}{1-t}\right).$$

Show that $u \in L^\infty(0,1;L^3)$ and that $u \notin C([0,1);L^3)$. Further, prove that $u(t,x) \rightharpoonup 0$ as $t \nearrow 1$.

EXERCISE 1.6 (*). *Prove that for initial data $u_0 \in L^3_\sigma \cap L^2$, the unique mild solution constructed by the contraction mapping principle satisfies $u_0 \in L^\infty(0,T;L^2)$.*

EXERCISE 1.7 (****). *Consider the 2D vorticity equations*

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega \tag{1.1a}$$

$$u = \nabla^\perp (\Delta)^{-1} \omega. \tag{1.1b}$$

Consider initial data $\omega(0) \in L^1(\mathbb{R}^2)$, which is critical (verify this). Sketch the proof that there exists an $\epsilon_0 > 0$ such that if $\|\omega(0)\|_{L^1} < \epsilon_0$ then there is a unique global mild solution to (1.1) which satisfies

$$\sup_{t \in (0,\infty)} t^{1/4} \|\omega(t)\|_{L^{4/3}} < \infty.$$

You will probably find the Hardy-Littlewood-Sobolev inequality helpful for getting L^p estimates on u in terms of the vorticity ω. Recall that the HLS is

$$\left\| f * |\cdot|^\alpha \right\|_{L^q(\mathbb{R}^d)} \lesssim_{p,q,d,\alpha} \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.2}$$

if $1 < p < q < \infty$, $0 < \alpha < d$ and

$$\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q} + 1.$$

Afterwards, try to extend this result to arbitrary data in L^1.

If you get lost, check out the original paper [**GMO88**].

EXERCISE 1.8 (***). *Prove the stronger version of the energy inequality for Leray-Hopf weak solutions, i.e. do Exercise 6.4 in Section 6 of the lecture notes.*

2. Inviscid limit with boundaries

EXERCISE 2.1 (**). *Consider the Euler and Navier-Stokes equations (with viscosity denoted by $\nu > 0$) on the half plane. Denote by (u, v, p) the Navier Stokes solution, and by $(\bar{u}, \bar{v}, \bar{p})$ the Euler solution. Denote by $\bar{U}(x,t)$ the trace of $\bar{u}$ at $y = 0$, that is*

$$\bar{U}(x,t) = \bar{u}(x,0,t).$$

Lastly, denote by $\Omega(x,t)$ the trace at $y = 0$ of the Navier-Stokes vorticity $\omega = \partial_x v - \partial_y u$, that is

$$\Omega(x,t) = \omega(x,0,t) = -(\partial_y u)(x,0,t).$$

Assuming that the initial data are smooth and obey

$$\lim_{\nu\to 0} \|u_0 - \bar{u}_0\|_{L^2} = 0$$

and that the traces of the solutions obey

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\mathbb{R}} U(x,t)\Omega(x,t)dxdt = 0$$

for some $T > 0$, prove that the inviscid limit in the kinetic energy norm

$$\sup_{t\in[0,T]} \|u(t) - \bar{u}(t)\|_{L^2} \to 0$$

holds.
Hint. *Look at the equation obeyed by $u - \bar{u}$, and perform an energy estimate.*

EXERCISE 2.2 (***). *Consider the two dimensional Navier-Stokes equation on the half plane, with anisotropic viscosity*

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \partial_{xx} u + \eta \partial_{yy} u \\ \nabla \cdot u &= 0 \\ u|_{y=0} &= 0. \end{aligned}$$

Consider also the two dimensional Euler equations on the half plane

$$\begin{aligned} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} &= 0 \\ \nabla \cdot \bar{u} &= 0 \\ \bar{u} \cdot n|_{y=0} &= 0. \end{aligned}$$

Consider initial datum of the Navier-Stokes and the Euler equations that are converging to each other, i.e.

$$\|u_0 - \bar{u}_0\|_{L^2} \to 0 \qquad as \qquad \nu, \eta \to 0,$$

and that they are smooth (say H^s with $s > d/2 + 1$). Let $T > 0$. Prove that in the limit

$$\nu \to 0, \qquad \eta \to 0, \qquad \frac{\eta}{\nu} \to 0$$

we have

$$\sup_{t\in[0,T]} \|u(t) - \bar{u}(t)\|_{L^2} \to 0,$$

i.e. we have convergence of the Navier-Stokes solution with anisotropic viscosity to the Euler solution, in the kinetic energy norm.
Hint. *Modify the corrector in the proof of Kato presented in class.*

3. Stability at high Reynolds number

EXERCISE 3.1 (*). *Do exercise 1.1 in the stability notes: Consider a finite dimensional ODE* $\partial_t X = AX$ *for a given fixed matrix* A *(and the equilibrium is of course* $f_0 = 0$*). Prove that if* A *is diagonalizable then spectral stability implies stability in the sense of Lyapunov. However, prove that if* A *is not diagonalizable, then spectral stability does* not *necessarily imply stability in the sense of Lyapunov.*

EXERCISE 3.2 (**). *Do exercise 1.2 in the stability notes: Consider a finite dimensional linear ODE* $\partial_t X = AX$ *for a given fixed matrix* A*. Prove that spectral instability always implies instability in the sense of Lyapunov (for the non-diagonalizable case, remember the Jordan block decomposition).*

EXERCISE 3.3 (**). *Consider a nonlinear finite dimensional ODE* $\partial_t X = F(X)$ *with equillibrium* $F(X_0) = 0$*. Show that if* $\nabla F(X_0)$ *is diagonalizable and* $\sigma(\nabla F(X_0)) \subset \{c \in \mathbb{C} : \textbf{Re}\, c < 0\}$*, then the equillibrium* X_0 *is nonlinearly (Lyapunov) stable. Give an example that shows nonlinear stablity can fail if we only assume* $\sigma(\nabla F(X_0)) \subset \{c \in \mathbb{C} : \textbf{Re}\, c \leq 0\}$.

EXERCISE 3.4 (**). *Do exercise 2.1 in the stability notes: For* $f_{in} \in L^2(\mathbb{T} \times \mathbb{R})$*, prove that if* f *solves* $\partial_t f + y\partial_x = 0$ *with* $f(0) = f_{in}$*, then* $f(t) \rightharpoonup \langle f_{in} \rangle_x$ *in* L^2*. Show that this convergence is only strong if* $f(t) = \langle f_{in} \rangle_x$ *for all* t*. Similarly, prove that if* $f_{in} \neq \langle f_{in} \rangle_x$ *and* $f_{in} \in H^n$*, then* $\|f(t)\|_{H^n} \approx \langle t \rangle^n \|f_{in}\|_{H^n}$ *(we are denoting* $\langle t \rangle = (1 + |t|^2)^{1/2}$*).*

References

[GMO88] Y. Giga, T. Miyakawa, and H. Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rat. Mech. Anal.*, 104(3):223–250, 1988.