THE ONSAGER CONJECTURE

R. SHVYDKOY

The Euler system governing evolution of an incompressible ideal homogeneous ($\rho \equiv 1$) fluid is given by

$$\text{(E)} \qquad \begin{cases} u_t + u \cdot \nabla u + \nabla p &= f, \quad f \text{ smooth} \\ \nabla \cdot u &= 0, \quad x \in \mathbb{T}^n \text{ or } \mathbb{R}^n \\ u(0) &= u_0. \end{cases}$$

Let $E(t) = \frac{1}{2}\int |u(x,t)|^2\,dx$. For classical solutions, we have the energy balance relation

$$\text{(EB)} \qquad E(t) = E_0 + \underbrace{\int_0^t f \cdot u\,ds.}_{\text{total work of force}}$$

The reason is that Flux $= \Pi = \int u \cdot \nabla u \cdot u\,dx = 0$.

Question: How much regularity is needed to still prove (EB)? If we want to approach this as in the classical sense, then we first need to learn how to control the flux. Heuristically,

$$\Pi \sim \int \nabla^{\frac{1}{3}} u \cdot \nabla^{\frac{1}{3}} u \cdot \nabla^{\frac{1}{3}} u\,dx = \int (\nabla^{\frac{1}{3}} u)^3\,dx.$$

So u has to be $\frac{1}{3}$-smooth in the proper sense.

Lars Onsager (version 1949):

(A) If u has smoothness better than $\frac{1}{3}$ (loosely for now), then (EB) holds.
(B) There exists a weak solution to (E) which is "$\frac{1}{3}$-smooth" and which violates (EB).

We have

$$E(t) = \underbrace{\int_0^T \Pi\,dt}_{\text{total energy flux}} + \underbrace{\int_0^T f \cdot u\,dt.}_{\text{work}}$$

If $[u] = U$, $[t] = T$, $[x] = L$, then the units of total flux are TU^3L^{n-1}. A function space X with $[X]^3 = TU^3L^{n-1}$ will be called *Onsager-critical*. For example, the following spaces are Onsager-critical:

$$\begin{array}{ll} L^3L^6 & \text{in 2D,} \\ L^3L^{9/2} \text{ and } L^3H^{\frac{5}{6}} & \text{in 3D,} \\ L^3W^{\frac{1}{3},3}, L^3B^{\frac{1}{3}}_{3,p}, 1 \le p \le \infty & \text{in any D} \end{array}$$

Exercise: Any Onsager-critical space which is L^p-based in time must be L^3_t.

Reference: Gregory L. Eyink and Katepalli R. Sreenivasan, *Onsager and the theory of hydrodynamic turbulence*, Rev. Modern Phys. **78** (2006), no. 1, 87–135. Contains a historic overview of Onsager's papers, etc.

Evidence from turbulence.

Kolmogorov proposed a theory in 1941 (K41) that postulates non-vanishing energy dissipation rate as a driving mechanism of turbulence. If u^ν satisfies

$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu = \nu \Delta u^\nu + f,$$

then we define the mean energy dissipation rate per unit mass by

$$\varepsilon^\nu = \nu \langle |\nabla u^\nu|^2 \rangle.$$

Here $\langle \cdot \rangle$ denotes ensemble average. Isotropic, homogeneous turbulence postulates that distribution laws of velocity increments (displacements), $\delta u(\ell) = u(r+\ell) - u(r)$, are independent of r and the direction of ℓ. The longitudinal structure functions $S_p(\ell) = \langle (\delta u(\ell) \cdot \frac{\ell}{|\ell|})^p \rangle$ can be measured experimentally.

Elements of K41 Theory:

- Kolmogorov's 0th law of turbulence: $\varepsilon^\nu \to \varepsilon > 0$ as $\nu \to 0$ (keeping L and U under control, i.e. $Re \to \infty$).
  We also have the following "provable" laws (assuming all formal manipulations are justified):
- Kolmogorov $\frac{4}{5}$-law (K-$\frac{4}{5}$): $S_3(\ell) = -\frac{4}{5}\varepsilon|\ell|$.
- Kármán-Howarth-Monin relation (KHM): $\Pi = -\frac{1}{4} \operatorname{div}_\ell \langle \delta u(\ell) |\delta u(\ell)|^2 \rangle \big|_{\ell=0}$.

The K-$\frac{4}{5}$ law clearly suggests that turbulence is generically $\frac{1}{3}$-regular. (It also suggests that regularity needs to be understood in an averaged sense, hence the need for Besov spaces below.) The KHM relation also implies that $\Pi \neq 0 \implies u$ is at most $\frac{1}{3}$-regular. Π and ε anomalies can be reconciled on a formal level.

Reference: Uriel Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995. The legacy of A. N. Kolmogorov.

Suppose $\langle \cdot \rangle$ is also stationary and $f = f_{<k_0}$, k_0 is an integral wavenumber, and $f_{<k_0}$ is the "filtered" field, i.e., the projection on the Fourier side. Let u solve the NSE. Then test with $u_{<k}$:

$$\underbrace{\frac{1}{2}\partial_t \langle |u_{<k}|^2 \rangle}_{=0 \text{ (stationary)}} + \underbrace{\langle (u \cdot \nabla u)_{<k} \cdot u_{<k} \rangle}_{=\Pi^\nu(k)} = -\nu \langle |\nabla u_{<k}|^2 \rangle + \underbrace{\langle f \cdot u_{<k} \rangle}_{\substack{=\langle f \cdot u \rangle, \\ \text{since } f = f_{<k_0}}}.$$

So $\Pi^\nu(k) = -\nu \langle |\nabla u_{<k}|^2 \rangle + \langle f \cdot u \rangle$. But

$$\nu \langle |\nabla u_{<k}|^2 \rangle \leq \nu \langle |u|^2 \rangle k^2 \sim \nu U^2 k^2 \xrightarrow{\nu \to 0} 0.$$

On the other hand, testing the NSE with the full u:

$$\underbrace{\partial_t \langle |u|^2 \rangle}_{=0 \text{ (stationary)}} + \underbrace{\langle u \cdot \nabla u \cdot u \rangle}_{=0 \text{ (regularity)}} = \underbrace{-\nu \langle |\nabla u|^2 \rangle}_{=-\varepsilon^\nu} + \langle f \cdot u \rangle,$$

and $\varepsilon^\nu = \langle f \cdot u^\nu \rangle \xrightarrow{\nu \to 0} \varepsilon$. So $\Pi(k) = \varepsilon$, for all $k > k_0$. Letting $k \to \infty$, conclude $\Pi = \varepsilon$.

This naïve computation also shows energy cascade: energy being fed by f dissipates from large to small scales with constant flux. See Figure 1.

The energy spectrum $E(k) = \frac{1}{2}\frac{d}{dk}\langle |u_{<k}|^2 \rangle$ distributes according to a scaling law.

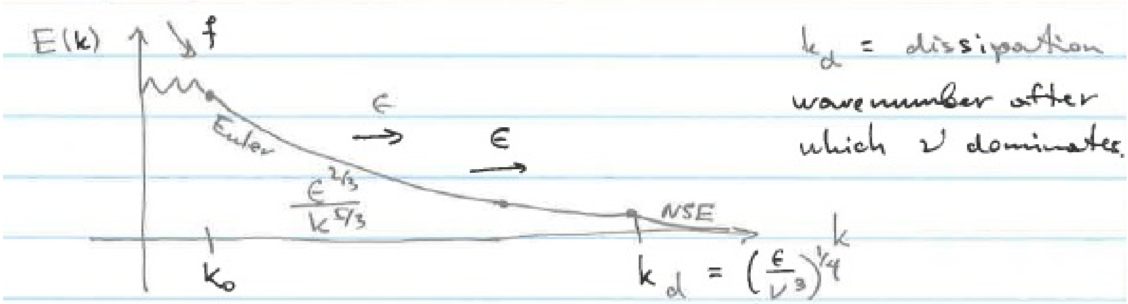

Figure 1.

- K41: All quantities depend only on ε and ℓ. Dimensional analysis implies that
$$E(k) = k_0 \frac{\varepsilon^{\frac{2}{3}}}{k^{\frac{5}{3}}}.$$
This famous law has been confirmed experimentally in many situations.
- Self-similarity: $\delta u(\ell) \sim \ell^h$. K-$\frac{4}{5}$ implies $h = \frac{1}{3}$, which in turn implies $S_p(\ell) = C_p \varepsilon^{\frac{p}{3}} \ell^{\frac{p}{3}}$. Deviations are observed but unexplained (see intermittency below).
- Locality of flux: $\Pi(k)$ is generated mostly by scales close to $\frac{1}{k}$.

Full confirmation of Onsager Conjecture for Burgers' Equation: Motivation for use of Besov Spaces.

Burgers' equation is

$$u_t + uu_x = 0. \tag{BE}$$

Suppose u_0 has a negative slope somewhere, so that a shock forms. (BE) represents the evolution of compressible fluid. After the shock, the solution becomes a weak solution (entropy solution), which describes the motion of particles after an inelastic collision (particles stick together and move with the same velocity).

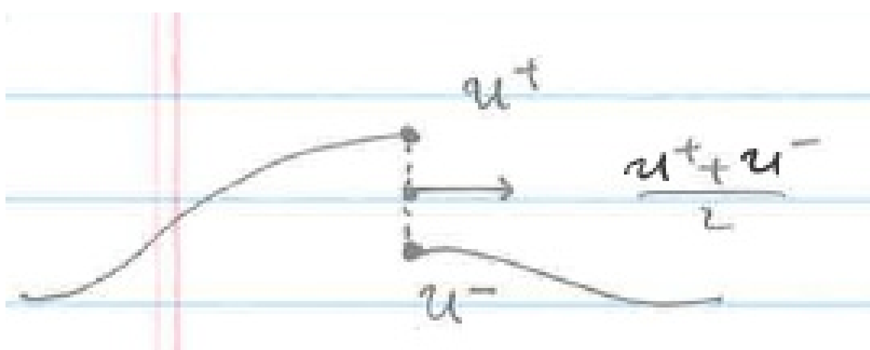

Figure 2.

Consider temporarily the collision of two particles, with velocities u^+ and u^-, respectively. Momentum is conserved: $u^+ + u^- = \frac{u^+ + u^-}{2} + \frac{u^+ + u^-}{2}$ before and after the collision. But $E_b =$ Energy before $= \frac{(u^+)^2 + (u^-)^2}{2}$, $E_a =$ Energy after $= \frac{1}{2}\left(\frac{u^+ + u^-}{2}\right)^2 + \frac{1}{2}\left(\frac{u^+ + u^-}{2}\right)^2 = \left(\frac{u^+ + u^-}{2}\right)^2$.

So $E_b > E_a$: energy is lost! The intuition from this simple calculation is reflected in the energy associated to the Burgers' equation. Indeed,

$$\frac{dE}{dt} = -\frac{1}{12}(u^+ - u^-)^3 < 0.$$

The proof of this statement is left as an exercise.

What regularity does a shock have? Certainly not Hölder! How do we measure smoothness?

$$\text{Hölder: } u \in C^s \ (= B^s_{\infty,\infty}) \text{ if } \sup_{y \neq 0} \sup_x \frac{|u(x+y) - u(x)|}{|y|^s} < \infty.$$

$$[\text{Replace } L^\infty_x \text{ by } L^p_x]$$

$$\text{Besov: } u \in B^s_{p,\infty} \text{ if } u \in L^p \text{ and } \sup_{y \neq 0} \int_{\mathbb{R}^n} \frac{|u(x+y) - u(x)|^p}{|y|^{ps}}\, dx < \infty.$$

The third index "∞" corresponds to L^∞_y and can be replaced by $p \rightsquigarrow L^p_y$ in a proper sense (not used here). Also,

$$u \in B^s_{p,c_0} \text{ if } u \in B^s_{p,\infty} \text{ and } \lim_{y \to 0} \int_{\mathbb{R}} \frac{|u(x+y) - u(x)|^p}{|y|^{ps}}\, dx = 0.$$

Besov functions have good approximation properties:

Lemma. *$u \in B^s_{p,\infty}$ ($1 \le p \le \infty$, $0 < s < 1$) iff $\|u_\delta - u\|_{L^p} \le C\delta^s$ as $\delta \to 0$, where u_δ is a standard mollification of u.*

Lemma. *A shock is in the class $B^{\frac{1}{p}}_{p,\infty}$ for all $1 \le p \le \infty$.*

(Proof left as an exercise. Hint: Check the Heaviside function.)

So, generic solutions to Burgers' equation start dissipating energy in finite time while remaining in the class $B^{\frac{1}{3}}_{3,\infty}$. On the other hand, we will show that in any dimension, if $u \in B^{\frac{1}{3}}_{3,c_0}$, then (EB) holds. This answers the Onsager conjecture for Burgers' equation completely.

Littlewood-Paley Theory.

Let $\chi = \varphi_{-1}$ be a decreasing radial C^∞ function supported in the unit ball, such that $\chi \equiv 1$ in $\{|\xi| \le \frac{1}{2}\}$. Define $\varphi_0 = \chi(\frac{\xi}{2}) - \chi(\xi)$ and $\varphi_q = \varphi_0(\frac{\xi}{2^q})$ for $q \ge 1$. Then $\sum_{q=-1}^\infty \varphi_q = 1$ pointwise. Let $u_q = (\varphi_q \widehat{u})^\vee$, so that $u = \sum_{q=-1}^\infty u_q$ in the sense of distributions. Let $u_{<q}$ denote the sum $u_{<q} = \sum_{p=-1}^{q-1} u_p$, and let $u_{\ge q} = u - u_{<q}$, etc.

Theorem (Littlewood-Paley)**.** *$u \in L^p$ iff $\left(\sum_{-1}^\infty |u_q|^2\right)^{\frac{1}{2}} = S(u) \in L^p$, and $\|u\|_{L^p} \sim \|S(u)\|_{L^p}$.*

Here $S(u)$ is called the "square function."

We will actually work with the dimensional dyadic wavenumber $\lambda_q = \frac{2^q}{L}$.

Lemma. *(i) $u \in B^s_{p,\infty}$ iff $\sup_q \lambda_q^s \|u_q\|_p < \infty$. (ii) $u \in B^s_{p,c_0}$ iff $\lim_{q \to 0} \sup_q \lambda_q^s \|u_q\|_p = 0$.*

The proof is left as an exercise.

We will be using time-averages (c.f. ensemble average) $\langle f \rangle = \frac{1}{T}\int_0^T f\, dt$.

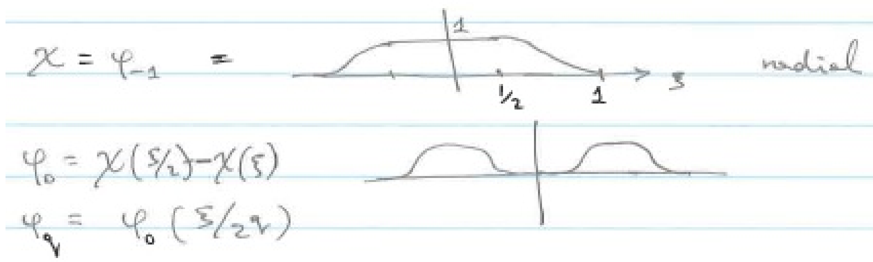

Figure 3.

Onsager Conjecture: Part (A)

Theorem. (Frisch-Sulem '75: $H^{\frac{5}{6}+\varepsilon}$; Eyink '94: $C^{\frac{1}{3}+\varepsilon}$; Constantin, E, Titi: $B^{\frac{1}{3}+\varepsilon}_{3,\infty}$; Duchon-Robert 2000: dissipation distribution, CET, local KMH; Cheskidov, Constantin, Friedlander, RS 2008: present version.) *Suppose $u \in C_w L^2_x$ is a weak solution to the Euler equations such that*

$$(*) \qquad \lim_{q\to\infty} \langle \lambda_q \|u_q\|_3^3 \rangle = 0 \quad \textit{on } [0,T].$$

Then the energy law (EB) *holds on the same interval.*

Remark. If $u \in L^3 B^{\frac{1}{3}}_{3,\infty}$, then $(*)$ holds. (Exercise. Hint: use Fatou's Lemma.)

Proof. The equation $\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0$ holds distributionally in $\mathcal{D}'$. Let us test with $(u_{<q})_{<q}$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{<q}\|^2_{L^2} = \int (u \otimes u)_{<q} : \nabla u_{<q}\, dx = \Pi_q.$$

Here $A : B = a_{ij} b_{ij}$. We have the identity

$$(u \otimes u)_{<q} = r_q(u,u) - u_{\geq q} \otimes u_{\geq q} + u_{<q} \otimes u_{<q},$$

where $r_q(u,u)(x) = \int h_q(y)\,(u(x+y) - u(x)) \otimes (u(x+y) - u(x))\; dy$, $h^q = \chi^\vee$, $h_q(y) = \lambda_q^n h(y\lambda_q)$. (Proof: Exercise.) So we are left with

$$\Pi_q = \underbrace{\int r_q(u,u) : \nabla u_{<q}\, dx}_{=:A} + \underbrace{\int u_{\geq q} \otimes u_{\geq q} : \nabla u_{<q}\, dx.}_{=:B}$$

We need the differential Bernstein inequality:

$$\|\nabla u_q\|_p \sim \lambda_q \|u_q\|_p, \quad 1 \leq p \leq \infty.$$

Let us first estimate B. We have

$$\begin{aligned}
|B| \leq \|u_{\geq q}\|_3^2 \|\nabla u_{<q}\|_3 &\leq \left(\sum_{p \geq q} \|u_p\|_3 \right)^2 \left(\sum_{p<q} \lambda_p \|u_p\|_3 \right) \\
&= \left(\sum_{p \geq q} \left(\frac{\lambda_q}{\lambda_p} \right)^{\frac{1}{3}} \lambda_p^{\frac{1}{3}} \|u_p\|_3 \right)^2 \left(\sum_{p<q} \left(\frac{\lambda_p}{\lambda_q} \right)^{\frac{2}{3}} \lambda_p^{\frac{1}{3}} \|u_p\|_3 \right) \\
&= \left(\sum_p K_{|p-q|} \lambda_p^{\frac{1}{3}} \|u_p\|_3 \right)^3 \qquad \text{(here } K_r = \lambda_{|r|}^{-\frac{1}{3}}\text{)} \\
&\leq \sum_p K_{|p-q|} \lambda_p \|u_p\|_3^3.
\end{aligned}$$

Now, let us estimate the term A. First, we have

$$|A| \leq \|\nabla u_{<q}\|_3 \|r_q(u,u)\|_{\frac{3}{2}}.$$

As before, $\|\nabla u_{<q}\|_3 \leq \sum_{p<q} \lambda_p \|u_p\|_3$. Next, we need the Minkowski inequality:
Suppose $f : \Omega \to X$ is a strongly measurable, Bochner integrable function. Then

$$\left\| \int_\Omega f \, d\mu \right\|_X \leq \int_\Omega \|f\|_X \, d\mu.$$

Continuing with the prove of the theorem, we have

$$\begin{aligned}
\|r_q(u,u)\|_{\frac{3}{2}} &= \left\| \int (u(\cdot + y) - u(\cdot)) \otimes (u(\cdot + y) - u(\cdot)) \underbrace{h_q(y)\, dy}_{d\mu} \right\|_{L^{\frac{3}{2}}} \\
&\leq \int \|u(\cdot + y) - u(\cdot)\|_{L^3}^2 h_q(y)\, dy \\
&\leq \int \|u_{<q}(\cdot + y) - u_{<q}(\cdot)\|_{L^3}^2 h_q(y)\, dy + \int \|u_{\geq q}\|_{L^3}^2 h_q(y)\, dy \\
&\leq \int_0^1 \int \|\nabla u_{<q}\|_{L^3}^2 \theta^2 |y|^2 \lambda_q^n h(\lambda_q y)\, dy\, d\theta + \int \left(\sum_{p \geq q} \|u_p\|_{L^3} \right)^2 h_q(y)\, dy \\
&\leq \left(\sum_{p<q} \lambda_p \|u_p\|_3 \right)^2 \frac{1}{\lambda_q^2} \underbrace{\int \lambda_q^n (\lambda_q |y|)^2 h(\lambda_q y)\, dy}_{= \text{const. independent of } q!} + \left(\sum_{p \geq q} \|u_p\|_3 \right)^2.
\end{aligned}$$

So

$$
\begin{aligned}
|A| &\le \left(\sum_{p<q} \lambda_p \|u_p\|_3\right)^2 \frac{1}{\lambda_q^2} + \left(\sum_{p<q} \lambda_p \|u_p\|_3\right)\left(\sum_{p\ge q} \|u_p\|_3\right)^2 \\
&= \left(\sum_{p<q} \left(\frac{\lambda_p}{\lambda_q}\right)^{\frac{2}{3}} \lambda_p^{\frac{1}{3}} \|u_p\|_3\right)^3 + \left(\sum_{p<q} \left(\frac{\lambda_p}{\lambda_q}\right)^{\frac{2}{3}} \lambda_p^{\frac{1}{3}} \|u_p\|_3\right)\left(\sum_{p\ge q} \left(\frac{\lambda_q}{\lambda_p}\right)^{\frac{1}{3}} \lambda_p^{\frac{1}{3}} \|u_p\|_3\right)^2 \\
&\le [\text{same as bound for } B].
\end{aligned}
$$

So $\langle \Pi_q \rangle \le C \sum_{p\in\mathbb{N}} K_{p-q} \langle \lambda_p \|u_p\|_3^3 \rangle$. This proves the theorem since $\langle \Pi_q \rangle \to 0$ and

$$
\begin{aligned}
\|u_{<q}(t)\|_{L^2}^2 - \|u_{<q}(0)\|_{L^2}^2 &= \langle \Pi_q \rangle + \langle f \cdot u_{<q} \rangle \\
\implies E(t) - E(0) &= \langle f \cdot u \rangle.
\end{aligned}
$$

It also provides a justification for locality of the energy flux in frequency space. □

Local Version

The local energy equality is a more detailed (in space) version of the law (EB) which is global in nature. Test (E) with φu, where $\varphi \in C_0^\infty$ is a scalar function. Then

$$
\begin{aligned}
\text{(LEB)}\quad & \int_{\mathbb{R}^n} |u(x,t)|^2 \varphi(x,t)\, dx - \int_{\mathbb{R}^n} |u(x,t_0)|^2 \varphi(x,t_0)\, dx - \int_{t_0}^t \int_{\mathbb{R}^n} |u(x,s)|^2 \partial_s \varphi(x,s)\, dx\, ds \\
&= \int_{t_0}^t \int_{\mathbb{R}^n} (|u|^2 + 2p) u \cdot \nabla \varphi \, dx\, ds + \int_{t_0}^t \int_{\mathbb{R}^n} (f \cdot u) \varphi \, dx\, ds.
\end{aligned}
$$

We define the global Onsager class $\mathcal{R}_0$ by

$$
u \in \mathcal{R}_0 \iff (*) \text{ holds},
$$

and for an open set $\Omega \subset \mathbb{R}^n \times [0,T]$, we say that $u \in \mathcal{R}_0(\Omega)$ iff $\varphi u \in \mathcal{R}_0$ for all $\varphi \in C_0^\infty(\Omega)$.

Theorem. *If $u \in \mathcal{R}_0(\Omega)$, then* (LEB) *holds for all $\varphi \in C_0^\infty(\Omega)$.*

Sister Conjectures

(1) Helicity $\int_{\mathbb{R}^3} u \cdot \omega \, dx$ requires $u \in B^{\frac{2}{3}}_{3,c_0} \cap H^{\frac{1}{2}}$. Helicity flux is given by (exercise)

$$
\frac{d}{dt} \int u_{<q} \cdot \omega_{<q} \, dx = \int (u \otimes u)_{<q} : \nabla \omega_{<q} \, dx + \int (u \times \omega)_{<q} \cdot \nabla u_{<q} \, dx.
$$

(2) Vorticity in 2D: $\|\omega\|_{L^2} =$ enstrophy conserved. A similar Littlewood-Paley argument requires $u \in B^1_{3,c_0}$ or $\omega \in B^0_{3,\infty} \cap L^2$. But $\omega_t + u \cdot \nabla \omega = 0$ is a transport equation! If we just assume $\omega \in L^2$ (which we do in order to make sense of the enstrophy), then $u \in W^{1,2} \hookrightarrow W^{1,1}$ which puts the solution into the framework of "renormalized" solutions in the sense of DiPerna-Lions. As a consequence, $\operatorname{meas}\{\omega > a\}$ is conserved, for all real a. Hence, $\|\omega\|_{L^2}^2 = 2\int_0^\infty \lambda \operatorname{meas}\{\omega > \lambda\}\, d\lambda$ is conserved. Lesson: The Onsager conjecture may still hold in Onsager-supercritical classes if there is extra structure of the equation (or solution) known.

(3) Similar conjectures can be stated for other advective systems, such as SQG, porous medium, etc.

Onsager Conjecture: Part (B)

In view of the result on the energy conservation, we know now that $B^{\frac{1}{3}}_{3,\infty}$ is critical. Let us now state the 2015 version of the Onsager Conjecture, part (B): Construct a pair (u, f) such that

(1) $u \in L^\infty_t L^2_x$, $f \in L^2$, $\sup_q \langle \lambda_q \|u_q\|_3^3 \rangle < \infty$.
(2) In the sense of distributions,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p &= f \\ \nabla \cdot u &= 0. \end{cases}$$

(3) There exists $t < T$ such that $E(t) \neq E(0) + \int_0^t f \cdot u \, ds$.

Remark. (1) $[L^3 B^{\frac{1}{3}}_{3,\infty}]^3 = TU^3 L^{n-1}$ in *any* dimension, including $n = 1$! (2) Note that $f \in L^2$ is the minimal assumption under which the pairing $\int f \cdot u \, dx$ can be understood classically. The predual to $B^{\frac{1}{3}}_{3,\infty}$ is $B^{-\frac{1}{3}}_{\frac{3}{2},1}$, which is weaker but is less physically reasonable for f to be in.

Timeline of attempts to solve part (B) of the Onsager Conjecture.

- Scheffer 1998: $u \in L^2_t L^2_x$ has compact support in time.
- Shnirelman 2000: same result, much shorter proof. Both constructions use backward cascade ideas.
- Convex integration era:

  DeLellis & Székelyhidi 2007 (Annals): $u \in L^\infty_t L^\infty_x$ even more striking; given any $\Omega \in \mathbb{R}^n \times (0, 1)$, we can have $|u| = X_\Omega$. There are extensions in the contexts of the active scalar equation, MHD, compressible, Perrono-MaCik, etc.

  DeLellis & Székelyhidi, $u \in C^{\frac{1}{10}-\varepsilon}_{t,x}$

  P. Isett, $u \in C^{\frac{1}{5}-\varepsilon}_{t,x}$

  Buckmaster, DeLellis, Székelyhidi, $u \in L^1_t C^{\frac{1}{3}-\varepsilon}_x$.

All these examples also demonstrate ill-posedness of the Euler system in the spaces where the examples live. It is also true for the Onsager-Besov spaces as well. Even more:

- C. Bardos, E. Titi '09 use an example of solution from DePerna-Majda '87 (Lions '98), namely $u(x, t) = \langle u_1(x_2), 0, u_3(x_1 - tu_1(x_2)) \rangle$, and show that there exist u_1, u_2 such that $u(0) \in C^\alpha$, $0 < \alpha < 1$, but $u(t) \notin C^{\alpha^2+\varepsilon}$ for any $\varepsilon > 0$.
- Cheskidov, RS '09: There exists $u_0 \in B^s_{p,\infty}$ for $s > 0$, $p > 2$ or $s > n(\frac{2}{p} - 1)$, $1 \leq p \leq 2$, such that any weak solution to (E) satisfies

$$\lim_{t \to 0^+} \|u(t) - u_0\|_{B^s_{p,\infty}} \geq \delta_0 > 0.$$

  In particular, in $B^{\frac{1}{3}}_{3,\infty}$, the trajectories are not continuous.

Example of a Non-zero Flux

There exists a field $u \in B^{\frac{1}{3}}_{3,\infty}$ such that $\limsup_{q\to\infty} \Pi_q > 0$.

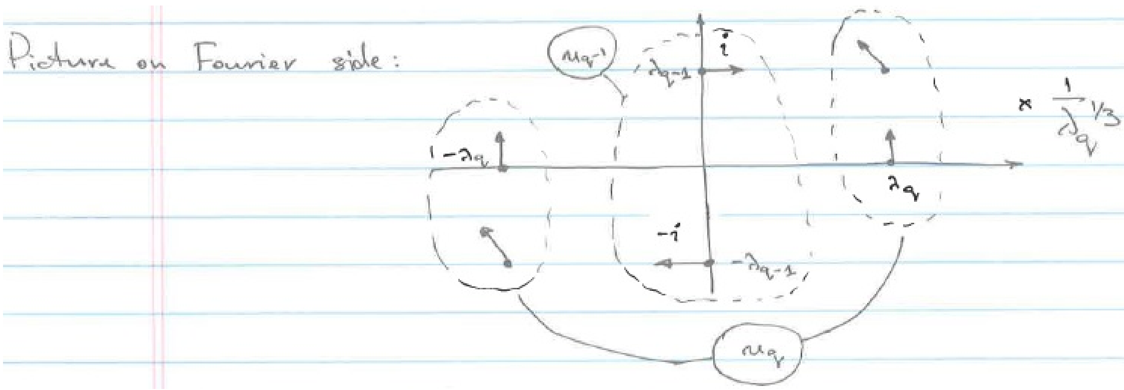

FIGURE 4.

Let $\widetilde{u}_1 = u_q + u_{q-1}$. Also,

$$\begin{aligned}
\widetilde{\Pi}_{q-1} &= \int (\widetilde{u}_q \otimes \widetilde{u}_q)_{\le q-1} : \nabla(\widetilde{u}_q)_{\le q-1}\, dx \\
&= \int (u_q + u_{q-1}) \cdot \nabla u_{q-1} \cdot (u_q + u_{q-1})\, dx \\
&= \int u_q \cdot \nabla u_{q-1} \cdot u_q\, dx \\
&= i \sum_{\xi} \sum_{\eta_1+\eta_2=\xi} (\widehat{u}_q(\eta_1)\cdot \xi)(\widehat{u}_{q-1}(\xi)\cdot \widehat{u}_q(\eta_2)) \\
&= \text{(by hand)} \ = c_0 > 0.
\end{aligned}$$

Let $u = \sum_{k=0}^{\infty} \widetilde{u}_{q_k}$, where $|q_k - q_{k-1}| = e^{e^k}$. Then $\Pi_{q-1} = \widetilde{\Pi}_{q-1} + \text{Res}$, where $\text{Res} \ll c_0$ due to outer locality estimates on the flux. So, $\limsup_{q\to\infty} \Pi_q \ge \frac{c_0}{2}$.

Looking ahead towards obtaining better regularity, can we improve $s = \frac{1}{3}$ by sacrificing $p = 3$ to a lower index? Tool: Sobolev embedding or Bernstein's inequalities. Note that in 2D,

$$\begin{aligned}
\operatorname{curl} u = \omega \in L^{\frac{3}{2}} \iff u \in W^{1,\frac{3}{2}} &\hookrightarrow W^{\frac{1}{3},3} \hookrightarrow B^{\frac{1}{3}}_{3,c_0} \\
&\implies \Pi_q = 0.
\end{aligned}$$

So, in 2D we can only improve ω to $L^{p<\frac{3}{2}}$.

Saturating Bernstein's Inequalities. Bernstein's inequality is the endpoint Sobolev inequality for a given dyadic shell: if $|\operatorname{supp}\widehat{u}| \subset [\lambda_{k-1}, \lambda_{k+1}]$, then $\|u\|_q \le \lambda_k^{n(\frac{1}{p}-\frac{1}{q})}\|u\|_p$ for all $1 \le p < q \le \infty$.

Proof. Let $h_k = \varphi_{k-1}^\vee + \varphi_k^\vee + \varphi_{k+1}^\vee$. Then $u = u * h_k$. By Young's inequality,

$$\|u\|_q = \|u * h_k\|_q \le \|u\|_p \|h_k\|_r \le C\|u\|_p \lambda_k^{n(1-\frac{1}{r})=n(\frac{1}{p}-\frac{1}{q})},$$

where $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$, so that $\|h_k\|_r^r = \int \lambda_k^{nr} |\varphi_0^\vee(\lambda_k x)|^r\, dx = C\lambda_k^{nr-n}$. $\square$

So on $\mathbb{T}^n$ we have the two inequalities $\|u\|_p \le \|u\|_q \le \lambda_k^{n(\frac{1}{p}-\frac{1}{q})}\|u\|_p$.

The lower bound prevails when u is "uniformly" distributed in x and "concentrated" to a point in ξ. The upper bound prevails when u is "uniformly" distributed in ξ and "concentrated" in x.

Example: $u(x) = \cos(\lambda_k x)$, all L^p-norms are similar; in complex variables, an even more vivid example is $u(x) = e^{i\lambda_k \cdot x}$. On the Fourier side, u has only one mode.

Example: The Dirichlet kernel $D_N(x) = \sum_{k=-N}^{N} e^{ikx}$ has modes which are uniformly distributed on the Fourier side, and D_N is concentrated near $x = 0$ on the x-side. Recall that

$$\|D_N\|_{L^p} \cong \begin{cases} N^{1-\frac{1}{p}}, & 1 < p \leq \infty \\ \log N, & p = 1. \end{cases}$$

Set $u(x) = e^{i\lambda_k \cdot x_1} D_{\frac{\lambda_k}{10}}(x_1) \cdot \ldots \cdot e^{i\lambda_k \cdot x_n} D_{\frac{\lambda_k}{10}}(x_n)$.

FIGURE 5.

We say in this case that the Bernstein inequality has been *saturated*. Let us now "saturate" our flux construction.

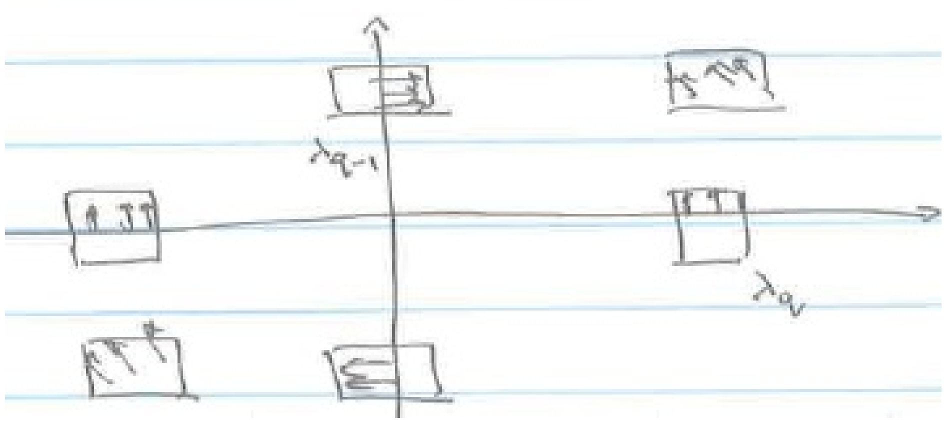

FIGURE 6.

Each block is scaled by $\frac{1}{\lambda_q^{\alpha}}$. What should α be? We have

$$\lambda_q^{\frac{1}{3}} \|u_q\|_3 = \lambda_q^{\frac{1}{3}-\alpha} \|D_{\lambda_q}\|_3^2 = \lambda_q^{\frac{4}{3}+\frac{1}{3}-\alpha=0}$$

if $\alpha = \frac{5}{3}$. But then for all $p < \frac{3}{2}$, $\|\nabla u_q\|_p = \lambda_q \|u_q\|_p \cong \lambda_q^{\frac{5}{3}-\frac{2}{p}} \|u_q\|_3 \cong \lambda_q^{\frac{4}{3}-\frac{2}{p}}$, and $\frac{4}{3} - \frac{2}{p} \leq 0$ iff $p \leq \frac{3}{2}$! So $u \in B^{\frac{1}{3}}_{3,\infty} \cap B^{1}_{\frac{3}{2},\infty} \implies \omega \in L^p$ for all $p < \frac{3}{2}$.

Mechanisms for Energy Balance Restoration in Onsager Critical and Supercritical Spaces

Transport.

(1) The DiPerna-Majda example $u = \langle u_1(x_2), 0, u_3(x_1 - tu_1(x_2))\rangle$ satisfies the Euler equation *without* pressure: $\partial_t u + u \cdot \nabla u = 0$. So each component of u is transported along a divergence-free field. For this solution, we clearly have $E(t) = E(0)$.

(2) $\omega \in L^2$ uses transport nature of vorticity in 2D to show enstrophy conservation in supercritical space.

(3) Exploring transport and the maximum principle for the Navier Stokes equations in 2D, we can obtain (EB) via the vanishing viscosity limit.

Theorem. (H. Nussenzveig-Lopes, M. Lopes, A. Cheskidov, RS 2015.) *Suppose $u_0 \in W^{1,p}$, $1 < p < \infty$ and hence $\omega_0 \in L^p(\mathbb{T}^2)$. Let us solve the NSE with initial datum ω_0:*

$$\begin{cases} \partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu &= \nu \Delta \omega^\nu \\ \omega^\nu(0) &= \omega_0. \end{cases}$$

There is a global in time solution with $\omega^\nu \in L^\infty_t L^p_x$.

Then *any* weak limit u of the sequence $\{u^\nu\}$ would solve the Euler equation and u would conserve energy, while being Onsager supercritical. Note again that in our flux example, $\omega \in L^p$ for all $p < \frac{3}{2}$. So to achieve the energy conservation, we have to *avoid* any attempt to bound the flux. Instead, we will show that the Kolmogorov 0th law fails.

Proof. We know that NSE in 2D is globally well-posed and that it regularized u instantly. This is more than enough to pass to the limit:

$$u^\nu_t + u^\nu \cdot \nabla u^\nu = -\nu \Delta u^\nu + \Delta p^\nu.$$

Take the dot product with v divergence-free to obtain

$$\int u^\nu(t) \cdot v \, dx - \int u_0 \cdot v \, dx - \int_0^t \int \underbrace{u^\nu \otimes u^\nu}_{\substack{\to u \otimes u \\ \text{strongly.}}} : \nabla v \, dx \, dt = -\nu \int_0^t \int u^\nu \Delta v \, dx \, dt.$$

Now test NSE for vorticities with ω^ν:

$$\frac{d}{dt} \|\omega^\nu\|^2_{L^2} = -2\nu \underbrace{\|\nabla \omega^\nu\|^2_{L^2}}_{\text{enthalpy}}.$$

Let us assume that $1 < p < 2$, for otherwise our general part (A) result covers it. By the Gagliardo-Nirenberg-Sobolev inequality,

$$\begin{aligned} \|\omega^\nu\|_{L^2} &\le \|\nabla \omega^\nu\|^{1-\frac{p}{2}}_{L^2} \|\omega\|^{\frac{p}{2}}_{L^p} \\ \implies -2\nu \|\nabla \omega^\nu\|^2_{L^2} &\le -2\nu \|\omega^\nu\|^{\frac{4}{2-p}}_{L^2} \|\omega\|^{-\frac{2p}{2-p}}_{L^p} \\ &\le -2\nu C_0 \|\omega^\nu\|^{\frac{4}{2-p}}_{L^2}. \end{aligned}$$

Here the last inequality is justified by the maximum principle: $\|\omega^\nu(t)\|_{L^p} \le \|\omega^\nu_0\|_{L^p}$.

Let $y(t) = \|\omega^\nu(t)\|^2_{L^2}$. We have the differential inequality

$$y' \le -C_0 \nu y^{\frac{2}{2-p}} \implies \frac{dy}{y^{\frac{2}{2-p}}} \le -c_0 \nu.$$

Integrate over $[\delta, t]$ to obtain

$$\frac{y(t)^{-\frac{p}{2-p}} - y(\delta)^{-\frac{p}{2-p}}}{-\frac{p}{2-p}} \leq -c_0\nu(t-\delta)$$

$$\implies \frac{1}{y(t)^{\frac{p}{2-p}}} - \frac{1}{y(\delta)^{\frac{p}{2-p}}} \geq c_p\nu(t-\delta).$$

We have assumed that $\omega_0^\nu \in L^p$ but not in $L^2 \implies y(\delta) \to \infty$ as $\delta \to 0$. Hence,

$$\|\omega^\nu(t)\|_{L^2}^2 = y(t) \leq \left[\frac{C}{\nu t}\right]^{\frac{2-p}{p}} = C(\nu t)^{\frac{p-2}{p}}.$$

Recall the energy inequality for solutions to the NSE:

$$\|u^\nu(t)\|_2^2 = \|u_0\|_2^2 - 2\nu \int_0^t \|\omega^\nu(s)\|_2^2 \, ds$$

$$\|u_0\|_2^2 - C\nu^{1-\frac{2-p}{p}} t^{1-\frac{2-p}{p}} = \|u_0\|_2^2 - C(\nu t)^{\frac{2p-2}{p}} \to \|u_0\|_2^2.$$

So in the limit as $\nu \to 0$, $\|u(t)\|_2^2 \geq \|u_0\|_2^2$. But also,

$$\|u^\vee(t)\|_2^2 + 2\nu \int_0^t \|\omega^\vee(s)\|_2^2 \, ds = \|u_0\|_2^2$$

$$\implies \|u(t)\|_2^2 \leq \|u_0\|_2^2.$$

□

Frozen Singularities. Vortex sheets in 2D or 3D are exactly Onsager-critical solutions.

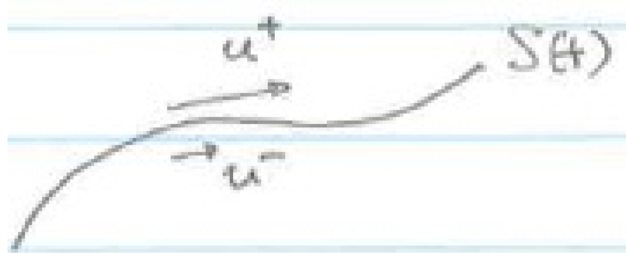

Figure 7.

In figure 7, u has a jump across $S(t)$, and $(u^\nu)^+ = (u^\nu)^-$ by the divergence-free condition. Also, $\omega = \gamma\delta_{S(t)}$, $\gamma = u_2^+ - u_2^-$. $S(t)$ is "frozen" into the flow, that is, it is transported by the average $U = \frac{u^+ + u^-}{2}$. This is dictated by the weak formulation of the Euler equations.

$$\Pi(t) = \int_{S(t)} (|u^+|^2 - |u^-|^2) \, d\mu_t,$$

where $d\mu_t = (\frac{\partial_t H}{|\nabla H|} + U \cdot n) \, d\sigma_t$, and $d\sigma_t$ is surface measure, $S(t) = \{H = 0\}$. But $d\mu_t = 0$ is the condition of being "frozen." The difference between Burgers' shock and the shock of the vortex sheet is that particles in the Burgers' case are allowed to "jump" into the shock, while in the vortex sheet they are not. So there is no exchange of energy between the two sides of the sheet.

Symmetries (Hamiltonian, scaling, etc). The Euler equation has a two-parameter family of scaling symmetries: if (u,p) solves (E), then the new pair

$$u_\lambda = \lambda^\alpha u(\lambda^\beta x, \lambda^{\alpha+\beta} t),$$
$$p_\lambda = \lambda^{2\alpha} p(\lambda^\beta x, \lambda^{\alpha+\beta} t)$$

solves the same equation (E). Solutions that are invariant under one of these symmetries are called *self-similar*; among them are stationary solutions. They are necessarily homogeneous:

$$u = \frac{U(\bar{x})}{|x|^\alpha} \qquad p = \frac{P(\bar{x})}{|x|^{2\alpha}}.$$

It is an interesting area of research to classify all such solutions (V. Sverak for 3D NSE, RS & Xue Luo for 2D Euler (arXiv)). In general, for $\alpha = \frac{n-1}{3}$, where n is the dimension of the space, the regularity of u at the origin becomes exactly Onsager-critical: $B^{\frac{1}{3}}_{3,\infty}$. Quick Check: Since u is $(-\alpha)$-homogeneous, its Fourier transform is $(-n+\alpha)$-homogeneous, i.e. $\widehat{u}(\xi) = \frac{\Omega(\bar{\xi})}{|\xi|^{n-\alpha}}$. In the dyadic shell $\lambda_{q-1} \leq |\xi| \leq \lambda_{q+1}$ all $|\xi|$s are comparable, and yet Ω is radially uniformly distributed. Also, Ω is smooth, so it changes smoothly on the sphere S^{n-1}. This creates saturation of Bernstein's inequalities:

$$\|u_q\|_3 \sim \frac{1}{\lambda^{n-\alpha}} \|D^{\alpha n}_{\lambda_q}\|_3 = \frac{1}{\lambda_q^{n-\alpha}} \lambda_q^{n(1-\frac{1}{3})} = \lambda_q^{\alpha - \frac{n}{3}} = \lambda_q^{\frac{n}{3} - \frac{1}{3} - \frac{n}{3}} = \frac{1}{\lambda_q^{\frac{1}{3}}}.$$

So indeed, $u \in \dot{B}^{\frac{1}{3}}_{3,\infty}$ (dot means "homogeneous"). In order to place u into L^3 we need to truncate it at ∞, which creates extra forcing in the equation. This will be done in 2D and 3D separately, as the construction relies on the existence of a stream-function or field.