Hydrodynamic stability and mixing at high Reynolds number: MSRI Graduate Summer school 2015

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1 Introduction

1.1 Hydrodynamic stability: a bit of history and context

Hydrodynamic stability is one of the oldest fields of fluid mechanics, dating back to the 1800s and attracting the attention of many greats such as Reynolds, Stokes, Lord Kelvin, Lord Rayleigh, and many others. This field is concerned with understanding the stability of *laminar* flow configurations and, specifically, describing when and how these flows become unstable; modern applications naturally include techniques for inhibiting, triggering, or otherwise controlling these instabilities. Aside from these questions being theoretically natural, they are also of practical relevance and, indeed, hydrodynamic stability is one of the main pillars of applied fluid mechanics. As we will see, it will also provide us an interesting weakly nonlinear regime to study some fundamental processes in fluid mechanics which are difficult to get a handle on otherwise.

The term *Reynolds number* is due to Reynolds' hydrodynamic stability experiments flow through a pipe. For a cylindrical flow configuration $(r, \theta, z) \in [0, R] \times [0, 2\pi) \times \mathbb{R}$, one can verify that

$$u_z = \frac{AR^2}{2\nu}\left(1 - \left(\frac{r}{R}\right)^2\right), \quad u_\theta = 0, \quad u_r = 0, \quad p = -Az$$

is a solution to the 3D Navier-Stokes equations; this configuration represents pressure driven flow in a pipe. What Reynolds did in [Rey83] was force fluid through a pipe while varying the various parameters and observed that for small **Re** this laminar flow is stable but that for sufficiently large **Re** he observed instability and spontaneous transition to turbulence. As theory and experiment progressed it became clear that this transition to turbulence was occurring in systems which are *spectrally stable* (see below)– this kind of transition is called *subcritical transition* or *by-pass transition.* This raised many interesting questions, especially since linear stability and nonlinear stability were almost always assumed to go together in applied mathematics at the time.

Since the work of Reynolds, countless experiments and computer simulations have been done on hydrodynamic stability problems in both 2D and 3D and the subject is both vast and rich; see e.g. the texts [DR81, Yag12, SH01, Dra02] and the references therein. We will discuss some of the many facets of the theory as we go along.

1.2 Notions of stability

Let N be a given (possibly nonlinear) operator N and suppose we have the abstract evolution equation

$$\partial_t f = N[f], \tag{1.1a}$$
$$f(0) = f_{in}, \tag{1.1b}$$

with the equilibrium point $N[f_0] = 0$. We will not trouble ourselves with well-posedness issues here, so we can assume the abstract system (1.1) is well-posed in whatever spaces we care about.

We will not be discussing spectral theory much since in fluid mechanics since most of the linear operators are non-normal, which means $AA^* \neq A^*A$. The spectral theorem shows that it is reasonable to think of this as the correct generalization of "non-diagonalizable" from basic linear algebra; see e.g. [RS79]. In particular, for non-normal operators, the spectrum of A may not tell us enough information about the linear evolution $\partial_t f = Af$. We will see some examples for finite dimensional linear ODEs below. Recall that the definition of spectrum for unbounded operators is the following; see e.g. [EN00, RS79].

Definition 1. Let H be a Hilbert space and $A : D(A) \to H$ be a closed operator with domain $D(A) \subset H$[1]. The *resolvent set* $\rho(A) \subset \mathbb{C}$ is given by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is invertible on } D(A) \to H\}.$$

The *spectrum* is then defined as $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Remark 1. In infinite dimensions, not everything in the spectrum corresponds to eigenvalues. We will be studying a very important linear operator $L = y\partial_x f$ on $\mathbb{T} \times \mathbb{R}$, which is skew-adjoint on $L^2(\mathbb{T} \times \mathbb{R})$.

Exercise 1.1. Verify that all eigenfunctions in L^2 of L are independent of x and correspond to the zero eigenvalue. That is, prove that if $f \in L^2$ and $y\partial_x f = \lambda f$, then $f(x, y) = \phi(y)$ for some ϕ (up to redefinition on a set of measure zero).

We see from Exercise 1.1 that $0 \in \sigma(L)$. However, let us verify that $i\lambda \in \sigma(L)$ for all $\lambda \in \mathbb{R}$, that is, the entire imaginary axis is also in the spectrum. For this, we need to ensure that the $i\lambda$ is not in the resolvent by proving that $L - i\lambda I$ is not invertible. We know already that we cannot do this by exhibiting an eigenfunction, however, we can do this by exhibiting a sequence of functions f_n such that

$$(L - i\lambda)f_n \to 0$$

but that $\|f_n\|_{L^2} \approx 1$.

Let ϕ be a smooth, compactly supported bump function. Fix a $k \in \mathbb{R}$ and consider now the sequence

$$f_n(x, y) = e^{ikx} n^{1/2} \phi\left(n(y - \frac{\lambda}{k})\right).$$

We can immediately check that $\|f_n\|_{L^2} = \sqrt{2\pi}\|\phi\|_{L^2}$, which is a fixed number. We can directly compute that

$$(L - i\lambda)f_n = (iky - i\lambda)f_n$$

However

$$\begin{aligned}
\|(iky - i\lambda)f_n\|_{L^2}^2 &= \int n\,|\lambda - ky|^2\,\phi^2\left(n(y - \lambda/k)\right)dxdy \\
&= \int n\,|kz|^2\,\phi^2\left(nz\right)dxdz \\
&= n^{-2}\int |kv|^2\,\phi^2(v)dxdv.
\end{aligned}$$

We see that this goes to zero as $n \to 0$ which shows that $L - i\lambda I$ is not invertible. Therefore, $i\lambda \in \sigma(L)$.

Regardless of all the potential issues with using the spectrum to assess stability, it is still a very useful notion, and one that is still the most common used by engineers and physicists.

[1] Recall that for unbounded operators, for example $A = -\Delta$ on $H = L^2$, we cannot make sense of the operator over the entire Hilbert space, hence we define the domain $D(A)$ as the subset for which it makes sense to consider $A : D(A) \to L^2$ (for example, if $A = -\Delta$ and $L^2(\mathbb{T}^2)$ then we should take $D(A) = H^2(\mathbb{T}^2)$.

Definition 2 (Spectral stability)**.** Let $Lf = DN[f_0]f$ be the linearization of N. The equilibrium f_0 is called *spectrally stable* in a Hilbert space X if $\sigma(L) \cap \{c \in \mathbb{C} : \mathrm{Re}\, c > 0\} = \emptyset$, where $\sigma(L)$ denotes the spectrum of L in X. The evolution is called *spectrally unstable* if $\sigma(L) \cap \{c \in \mathbb{C} : \mathrm{Re}\, c > 0\} \neq \emptyset$.

Remark 2. Often times, if system is spectrally stable but the spectrum intersects the imaginary axis, the terminology *neutrally stable* is used.

A different notion of stability, which is usually attributed to Lyapunov, is the following. When a mathematician says "stable", this is normally what he or she means.

Definition 3 (Lyapunov/nonlinear stability)**.** Given two Banach spaces X and Y (usually the same), the equilibrium f_0 is called *stable* (from X to Y) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that if

$$\|f_{in} - f_0\|_X < \delta.$$

then for all $t > 0$, the solution to (1.1) satisfies

$$\|f(t) - f_0\|_Y < \epsilon.$$

We say f_0 is *unstable* if it is not stable.

Remark 3. In many applications, $X = Y$, however in some applications this is too much to ask and we instead take X to be a smaller space than Y (e.g. the restriction on the initial data is stronger than the norm in which stability is deduced).

Even for *finite dimensional **linear** systems*, these two definitions of stability are *not* equivalent.

Exercise 1.2. Consider a finite dimensional ODE $\partial_t X = AX$ for a given fixed matrix A (and the equilibrium is of course $f_0 = 0$). Prove that if A is diagonalizable then spectral stability implies Lyaponuv stability. However, prove that if A is not diagonalizable, then spectral stability does *not* necessarily imply Lyapunov stability.

Exercise 1.3. Consider a finite dimensional ODE $\partial_t X = AX$ for a given fixed matrix A. Prove that spectral instability always implies instability in the sense of Lyapunov.

Exercise 1.4. Consider a nonlinear finite dimensional ODE $\partial_t X = F(X)$ with equilibrium $F(X_0) = 0$. Show that if $\nabla F(X_0)$ is diagonalizable and $\sigma(\nabla F(X_0)) \subset \{c \in \mathbb{C} : \mathbf{Re}\, c < 0\}$, then the equilibrium X_0 is Lyapunov stable. Give an example that shows nonlinear stability can fail if we only assume $\sigma(\nabla F(X_0)) \subset \{c \in \mathbb{C} : \mathbf{Re}\, c \leq 0\}$ (hence, even for diagonalizable systems, neutral stability does not imply Lyapunov stability).

A good rule of thumb is that spectral stability is in some sense the *weakest* kind of stability but spectral *instability* tends to be a pretty strong kind of instability, and in many settings, spectral instability is enough to deduce nonlinear instability.

We would also like to mention that in infinite dimensions, stability depends a lot on the norms that you are measuring. For example, consider the (very relevant) PDE:

$$\partial_t f + y\partial_x f = 0 \tag{1.2a}$$

$$f(0) = f_{in}. \tag{1.2b}$$

Exercise 1.5. Prove that the the equilibrium $f \equiv 0$ for (1.2) is spectrally stable in L^2, Lyapunov stable in L^2, but Lyapunov unstable in H^1 (in fact any H^s with $s > 0$).

2 2D inviscid planar shear flows and Rayleigh's theorem

We begin our study of hydrodynamic stability of inviscid shear flows, the simplest of all flows:

$$u(x,y) = \begin{pmatrix} U(y) \\ 0 \end{pmatrix},$$

where we will assume that $U(y) \in C^\infty$. We want to investigate the stability of this configuration in the 2D Euler equations to start with. Later we will consider 3D and finite Reynolds number generalizations.

One of the first works on the theoretical side of hydrodynamic stability was that of Lord Rayleigh who in 1880 attempted to determine when a certain inviscid shear flow is stable or not in the 2D Euler equations [Ray80]. He was interested in *spectral* stability, that is, to determine when there does or does not exist unstable eigenvalues to the linearized problem. For the following, we will mostly follow the treatment in [Dra02]. Lord Rayleigh derived a *necessary* condition for spectral instability of 2D shear flows by what is sometimes referred to as the normal mode method. This is just a name for the method of looking for sets of orthogonal eigenfunctions and eigenvalues for the linearized problem. Rayleigh considered the linearization of 2D Euler about a given shear flow $(U(y), 0)$:

$$u_t + U(y)\partial_x u + \begin{pmatrix} u_2 U'(y) \\ 0 \end{pmatrix} = -\nabla p \tag{2.1a}$$

$$-\Delta p = 2U' \partial_x u_2 \tag{2.1b}$$

$$u \cdot n = 0. \tag{2.1c}$$

say for $y \in [-1, 1]$, a bounded channel.

Exercise 2.1. Write (2.1) as $\partial_t u + Lu = 0$ for a linear operator L and verify that $L^* L \neq LL^*$ (and hence L is non-normal).

In vorticity form, this becomes

$$\omega_t + U(y)\partial_x \omega - U'' \partial_x \psi = 0$$

$$\Delta \psi = \omega.$$

Notice that the zero-th Fourier mode in x $\omega_0(t,y) = \hat{\omega}(t,0,y) = \frac{1}{2\pi}\int \omega(t,x,y)dx$ is conserved by the evolution, so we may assume without loss of generality that $\int \omega_{in}(x,y)dx = 0$ and hence that remains true forward (and backward) in time (note that this conservation law is not true for the nonlinear problem). We re-write on the streamfunction:

$$\left(\partial_t + U(y)\partial_x\right)\Delta\psi - U''(y)\partial_x \psi = 0. \tag{2.2}$$

Note that if we are in a bounded channel, this comes with boundary conditions, in particular $\nabla\psi \cdot \tau = 0$ on the upper and lower boundaries.

The problem is still translation invariant with respect to x so we can use the Fourier transform in this direction. Therefore, we look for a solution of the form:

$$\psi(t,x,y) = \phi(y) e^{i\alpha(x-ct)}, \tag{2.3}$$

where $\alpha \in \mathbb{R}$ is non-zero and $c \in \mathbb{C}$. Notice that if (2.3) is going to satisfy the boundary condition $\nabla\psi \cdot \tau = 0$, we are going to need to impose $\phi|_{\pm 1} = 0$. Upon substitution of (2.3) into (2.2) we get *Rayleigh's problem*:

$$(U(y) - c)\left(\phi'' - \alpha^2\phi\right) - U''(y)\phi = 0. \tag{2.4}$$

We will have an unstable mode if we can find ϕ, c and α which solve (2.4) with Im$c \neq 0$ (noting that if α, c, ϕ is a solution, then so is $-\alpha, c, \phi$). The horrible degeneracy at the point $U(y) = c$ is known as the "critical layer" in the applied literature; this is only possible when $c \in \mathbb{R}$ and so is not relevant when we are looking for unstable. The degeneracy is connected with the continuous spectrum which lies on the imaginary axis. However, let us not be too concerned with this (yet).

Theorem 2.1 (Rayleigh [Ray80]). *Consider the 2D Euler equations linearized around the shear flow $(U(y), 0)$ with $y \in [-1, 1]$ (or $y \in \mathbb{R}$) given in* (2.1)*. If the linearized 2D Euler equations have an unstable eigenmode in H^1, then U'' must vanish at least at one point (hence, any flow without an inflection point is spectrally stable).*

Proof. The proof proceeds by proving that if there is an unstable eigenvalue, then necessarily there is an inflection point somewhere in the flow. This can be proved by an energy-type estimate on (2.4). If there is an unstable eigenvalue, then there is a solution to (2.4) with $\alpha \geq 0$, Im$c > 0$, and ϕ non-trivial. Dividing (2.4) by $U - c$ (note that it is non-vanishing because Im$c \neq 0$ and $U \in \mathbb{R}$) and multiplying by $\overline{\phi}$ an integrating by parts gives (note the boundary terms vanish due to $\phi|_{y=\pm 1} = 0$),

$$\int \left|\phi'\right|^2 + \alpha^2 \left|\phi\right|^2 dy + \int \frac{U''(y)}{U(y) - c} \left|\phi\right|^2 dy = 0.$$

Taking the imaginary part leaves us

$$\operatorname{Im} c \int \frac{U''(y)}{\left|U(y) - c\right|^2} \left|\phi\right|^2 dy = 0.$$

By assumption $\operatorname{Im} c > 0$, therefore we have,

$$\int \frac{U''(y)}{\left|U(y) - c\right|^2} \left|\phi\right|^2 dy = 0,$$

is a necessary condition for instability. This requires that $U''(y) = 0$ in at least one place, and hence the theorem follows. □

Remark 4. By elliptic regularity, if $\operatorname{Im} c \neq 0$, any H^1 solution to (2.4) will be C^∞.

There are sharper spectral stability conditions [Fjo50], however, to our knowledge, there is still no known sharp or nearly sharp condition for spectral stability. For example, the *Couette flow* $u = (y, 0)$ is linearly stable (as we will see) and so are flows that are nearby in a certain sense (see below).

As it turns out, the result of Lord Rayleigh unfortunately extends to 3D via a result known as *Squire's theorem* [Squ33]. This theorem shows that if the 3D planar shear flow has unstable eigenvalues, then so does the 2D problem, and hence if one is looking for eigenvalue instabilities, then studying 2D is sufficient in the sense that any planar shear flow which is spectrally stable in 2D is also spectrally stable in 3D. See e.g. [Dra02] for a proof (it is not too hard actually, you could do it as an exercise). I say "unfortunately" because this theorem is *horrifically* misleading as it suggests all interesting aspects of hydrodynamic stability can be found in 2D equations – this is *extremely false.* In fact, it can be shown[2] that pretty much *every* non-trivial shear flow is unstable in the sense of Lyapunov for the linearized 3D Euler equations in $(x, y, z) \in \mathbb{T} \times \mathbb{R}^2$!

Theorem 2.2 (Squire's theorem [Squ33])**.** *Consider the 3D linearized Euler equations near a planar shear flow $(U(y), 0, 0)$ between parallel plates $[-1, 1]$. If there is a 3D unstable mode, then there is a 2D unstable mode with a faster or equally as fast growth rate. As a consequence, any planar shear flow which is spectrally stable in the linear 2D Euler equations is spectrally stable also in the linear 3D Euler equations.*

The method of normal modes sheds light on many classical problems in hydrodynamic stability (and many other questions in hundreds of other fields), such as understanding the Rayleigh-Benard convection cells, the Kelvin-Helmholtz instability, the Rayleigh-Taylor instability, the instability of jets, and more; see e.g. [DR81, Dra02]. Unfortunately (its quite fortunate in another sense) there are two major shortcomings of this method for hydrodynamic stability at high Reynolds number: (A) the linearized inviscid problems usually do not have a purely discrete spectrum and so looking for eigenfunctions is not sufficient[3] and (B) the linearized problems are usually not normal operators and so knowing the spectrum is usually not sufficient. Recall a linear operator is normal if $AA^\star = A^\star A$ in a suitable sense (let us not dwell on the finer points of spectral and unbounded operator theory here). It is precisely this class of operators such that nice versions of the spectral mapping theorems hold. Recall, spectral mapping theorems roughly tell you that if you know the spectrum of A (and you have several technical conditions satisfied) then you not only know the spectrum of e^{tA} but you also know the norm of this semigroup (see e.g. [EN00] for more precise details). This is not true of non-normal operators: if A is not normal, than the norm of e^{tA} can vary wildly from what its spectrum suggests [TE05]. The simplest example is the ODE (which actually is pretty relevant to hydrodynamic stability as we will see):

$$\partial_t X = \begin{pmatrix} -\epsilon & 1 \\ 0 & -\epsilon \end{pmatrix}.$$

The eigenvalues of the matrix are $-\epsilon < 0$, hence if the operator were normal, the solutions would be decaying and the operator norm of e^{tA} would be uniformly bounded in ϵ. However, instead we have $\|X(t)\| \approx \|X_0\| \langle t \rangle e^{-\epsilon t} \lesssim \epsilon^{-1} \|X_0\|$, which shows that solutions can undergo a large transient growth before eventually decaying. It was Orr [Orr07] who, to our knowledge, first pointed out the potentially pivotal importance of non-normal transient growth in fluid mechanics. We will see several examples of linear (and nonlinear) problems in fluid mechanics which can undergo a large transient growth like the above.

3 Arnold's nonlinear stability theorem for shear flows in a channel

Due to the non-normal nature of the linearizations, one can worry that *nonlinear* stability at high or infinite Reynolds number can be hard to come by. This is because transient linear growth could carry small perturbations out of the linear regime and into the fully nonlinear regime, triggering a secondary instability, as first suggested by Orr [Orr07] (we will return to this idea later, which is by now classical in applied fluid mechanics, see e.g. [TTRD93, RSBH98, SH01, TE05, Yag12]). However, there is a beautiful and classical result of Arnold [Arn65] which uses a variational method

[2] See future lectures, or do it as an exercise by taking x averages of the 3D analogue of (2.1) and considering the PDE that is left over.

[3] The viscous problem may have a discrete spectrum (depending on the boundary conditions or functions spaces) however in the inviscid limit it will develop a continuous spectrum.

to provide a simple proof of Lyapunov stability (in the H^1 norm of u) for a class of 2D shear flows in the nonlinear Euler equations. The basic ideas also generalize to many other situations, see e.g. the review by [HMRW85] and the references therein.

To make life simple, let us explain the idea of Arnold in the channel $(x, y) \in \mathbb{T} \times [-1, 1]$ with no-penetration boundaries on the top and bottom edges; much more generality is possible [HMRW85]. Notice that this domain is *not simply connected* which means we have to be a bit careful about the vorticity-streamfunction formulation of the equations.

3.0.1 Conservation laws and vorticity-streamfunction formulation in the channel $\mathbb{T} \times [-1, 1]$

We consider smooth solutions to the 2D Euler equations

$$\partial_t u + u \cdot \nabla u = -\nabla p \tag{3.1a}$$
$$\nabla \cdot u = 0 \tag{3.1b}$$
$$u \cdot n|_{y=\pm 1} = u_2|_{y=\pm 1} = 0. \tag{3.1c}$$

The 2D incompressible Euler equations can be thought of formally as a Hamiltonian system for the kinetic energy if interpreted correctly (also due to Arnold originally [Arn66]). Recall that the energy is

$$E[u] = \frac{1}{2} \int |u|^2 \, dx.$$

Since we are in 2D, for any smooth function Φ, the associated *Casimir*,

$$C_\Phi[\omega] = \int \Phi(\omega) dx$$

is conserved, where $\omega = \partial_x u_2 - \partial_y u_1$ is the (scalar) vorticity. These Casimirs provide a useful infinite set of conservation laws, which is one of the reasons that the 2D Euler equations are very different than the 3D Euler equations.

Next, due to the Kelvin circulation theorem, the circulation around every connected component of the boundary is constant. That is,

$$\Gamma_i = \int_{\partial D_i} u(t) \cdot ds = \int_{\partial D_i} u(0) \cdot ds.$$

This is a general fact, in this case, it applies to the lines $y = \pm 1$. Due to the shape of the domain (in particular, the translation invariance in x), the total x momentum is still conserved. That is,

$$M_x = \int_{-1}^{1} \int u_1(t, x, y) dx dy = \int_{-1}^{1} \int u_1(0, x, y) dx dy.$$

Before continuing, let us briefly discuss the vorticity stream-function formulation in the domain $\mathbb{T} \times [-1, 1]$. Let $(0, 0, \omega) = \nabla \times u$ be the scalar vorticity. Next, consider looking for a streamfunction ψ which satisfies $\Delta \psi = \omega$ and $u = \nabla^\perp \psi$. In general, we know that since $u \cdot n|_{\partial D_i} = \nabla^\perp \psi \cdot n|_{\partial D_i} = \nabla \psi \cdot \tau|_{\partial D_i}$, ψ is constant on each connected component of the boundary. The streamfunction is determined only up to a constant, but since there are two disconnected pieces of the boundary, the difference between the constants associated with each boundary is not determined. In this case it is easy to see what the difference is. Consider taking x averages of u_1:

$$\begin{aligned} u_1(t,x,y) &= -\partial_y \psi(t,x,y) \\ \langle u_1 \rangle_x(t,y) &= -\partial_y \langle \psi \rangle_x(t,y). \end{aligned}$$

Integrating this in y gives

$$\frac{1}{2\pi} M_x = \frac{1}{2\pi} \int u_1(t,x,y) dx dy = \langle \psi \rangle_x(t,-1) - \langle \psi \rangle_x(t,1).$$

The LHS is a fixed number in time (the mean flow across the torus) and so the difference between the value of the streamfunction at the top and bottom is this constant. Without loss of generality we may as well take

$$\begin{aligned} \langle \psi \rangle_x(t,-1) &= 0 \\ \langle \psi \rangle_x(t,1) &= -\frac{1}{2\pi} M_x. \end{aligned}$$

Hence, to find the streamfunction given the vorticity, we can solve the Dirichlet problem

$$\begin{aligned} \Delta \psi &= \omega \\ \psi(x,-1) &= 0 \\ \psi(x,1) &= -\frac{1}{2\pi} M_x. \end{aligned}$$

Finally, let us note that when viewed in terms of the vorticity and streamfunction, (ω, ψ) the energy becomes

$$\begin{aligned} E[\omega] &= \frac{1}{2} \int \left| \nabla^{\perp} \psi \right|^2 dx \\ &= \frac{1}{2} \int |\nabla \psi|^2 dx \\ &= \frac{1}{2} \sum_i \int_{\partial D_i} \psi \nabla \psi \cdot n ds - \frac{1}{2} \int \psi \omega dx. \end{aligned}$$

Note that because ψ is constant along the boundaries, this becomes

$$E[\omega] = \frac{1}{2} \sum_i \psi|_{\partial D_i} \int_{\partial D_i} u \cdot ds - \frac{1}{2} \int \psi \omega dx.$$

Hence, $-\frac{1}{2} \int \psi \omega dx$ is conserved, since the energy, the circulations at the top and bottom, and the value of $\psi|_{\partial D_i}$ are all individually conserved.

3.0.2 Variational stability

The general scheme of Arnold is to find suitable Φ and a_i such that the equilibrium, ω_E, is a critical point of the conserved energy functional

$$H_C[\omega] = \frac{1}{2} \int |u|^2 dx + \int \Phi(\omega) dx + \sum_{\partial D_i} a_i \Gamma_i$$

and that H_C is locally convex and suitably coercive with respect to the L^2 norms of the velocity and vorticity in a neighborhood of the critical point. Recall, coercive with respect to a norm means that control on the energy functional controls the norm; see below.

First let us compute the first variation of H_C at ω_E due to perturbations ω, ψ. The perturbations are assumed to preserve the mean flow rate, and therefore $\psi(x,-1) = \psi(x,1) = 0$. Computing the first variation and integrating by parts (using $u_E = \nabla^\perp \psi_E$ and $u = \nabla^\perp \psi$) gives

$$\begin{aligned} DH_C[\omega_E]\omega &= \int u_E \cdot u dx + \int \Phi'(\omega_E)\omega dx + \sum_{\partial D_i} a_i \int_{\partial D_i} u \cdot ds \\ &= \int \nabla^\perp \psi_E \cdot \nabla^\perp \psi dx + \int \Phi'(\omega_E)\omega dx + \sum_{\partial D_i} a_i \int_{\partial D_i} u \cdot ds \\ &= \int \nabla \psi_E \cdot \nabla \psi dx + \int \Phi'(\omega_E)\omega dx + \sum_{\partial D_i} a_i \int_{\partial D_i} u \cdot ds \\ &= \int \psi_E \nabla \psi \cdot n ds - \int \psi_E \omega dx + \int \Phi'(\omega_E)\omega dx + \sum_{\partial D_i} a_i \int_{\partial D_i} u \cdot ds \\ &= -\int \psi_E \omega dx + \int \Phi'(\omega_E)\omega dx + \sum_{\partial D_i} (a_i - \psi_E|_{\partial D_i}) \int_{\partial D_i} u \cdot ds. \end{aligned}$$

In the last line we used that $\nabla\psi \cdot n = u \cdot \tau$ and that $\psi_E|_{\partial D_i}$ is constant along the boundaries. Hence, ω_E will be a critical point of H_C as soon as

$$\psi_E = \Phi'(\omega_E), \tag{3.2a}$$
$$a_i = \psi_E|_{\partial D_i}. \tag{3.2b}$$

Suppose that we have a functional relationship $\psi_E = \Psi(\omega_E)$. In this case, it suffices to have

$$\Psi(\omega_E) = \Phi'(\omega_E),$$

and hence we can then take, for some constant λ,

$$\Phi(t) = \int_0^t \Psi(\tau) d\tau + \lambda.$$

We see that the equilibrium determines Φ; now it suffices to see what kind of equillibria are such that H_C is convex near ω_E. Let $\omega_E(x) + \omega(t,x)$ solve the full nonlinear 2D Euler equations in the periodic strip. Consider now the conserved functional (the first term is conserved, last term is zero by construction, and the second to last term is a constant in time):

$$F[\omega] = H_C[\omega + \omega_E] - H_C[\omega_E] - DH_C[\omega_E]\omega.$$

Computing this out gives

$$F[\omega] = \int \frac{1}{2} |u|^2 dx + \int \Phi(\omega_E + \omega) - \Phi(\omega_E) - \Phi'(\omega_E)\omega dx.$$

If there exists a constant $\delta > 0$ such that $\Phi'' \geq \delta$, then Φ is uniformly convex and therefore

$$\int \Phi(\omega_E + \omega) - \Phi(\omega_E) - \Phi'(\omega_E)\omega dx \geq \frac{\delta}{2} \int |\omega|^2 dx.$$

In this case we have

$$F[\omega(0)] = F[\omega(t)] \geq \frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{\delta}{2}\|\omega(t)\|_{L^2}^2.$$

This means that the kinetic energy and the enstrophy of the perturbation are uniformly bounded. If there is a constant $C > 0$ such that $\Phi'' \leq C$ then we also get

$$F[\omega(0)] \leq \frac{1}{2}\|u(0)\|_{L^2}^2 + \frac{C}{2}\|\omega(0)\|_{L^2}^2,$$

which will imply the desired global nonlinear stability in L^2 of the velocity and vorticity (sometimes called the "energy norm" and "enstrophy norm"). Due to the divergence free condition, this is equivalent to the H^1 norm on the velocity.

The general procedure can be extended to some more general domains and other equillibria which satisfy some kind of functional relationship $\psi_E = \Psi(\omega_E)$ for some Ψ, however, let us continue to just think about what this means for shear flows. In the case of a shear flow $u_E = (U(y), 0)$,

$$\begin{aligned} \omega_E(y) &= -U'(y) \\ U(y) &= -\partial_y \psi(y). \end{aligned}$$

Therefore, taking y derivatives of (3.2a) gives

$$-U(y) = -\Phi''(\omega_E(y))U''(y),$$

or

$$\frac{U(y)}{U''(y)} = \Phi''(\omega_E(y)).$$

Putting everything together, we have proved the following beautiful theorem.

Theorem 3.1 (Arnold's nonlinear stability). *Let $u_E = (U(y), 0)$ be a shear flow on $\mathbb{T} \times [-1, 1]$ such that U is smooth and such that there is some smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ such that $\psi_E = \Psi(\omega_E)$ (where $\omega_E = -U'$ is the vorticity). Suppose (up to Galilean invariance) that there exists constants $\infty > C > \delta > 0$ which satisfy*

$$C > \frac{U(y)}{U''(y)} \geq \delta.$$

Then the equilibrium u_E is globally nonlinearly stable for the 2D incompressible Euler equations in the energy and enstrophy norms if we restrict to perturbations which conserve M_x the mean zero momentum. That is, if $u + u_E$ solves the 2D Euler equations in $\mathbb{T} \times [-1, 1]$ and $\omega = \partial_x u_2 - \partial_y u_1$ and satisfies $\int_{\mathbb{T}\times\mathbb{R}} u_1 dx = 0$, then there holds uniformly forward and backward in time (note that it is $u + u_E$ which solves the 2D Euler equations):

$$\|u(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \lesssim \frac{C}{\delta}\left(\|u(0)\|_{L^2}^2 + \|\omega(0)\|_{L^2}^2\right). \tag{3.3}$$

Remark 5. Due to the divergence free condition, (3.3) is equivalent to

$$\|u(t)\|_{H^1} \lesssim \frac{C}{\delta}\|u(0)\|_{H^1}.$$

Remark 6. The above theorem proves, for example, that the equilibrium $U(y) = 2 + y^2$ is nonlinearly stable in energy and enstrophy norms in the 2D Euler equations.

Remark 7. We will see later that it would be unreasonable to expect L^2 stability at the level of the velocity only, that is, we will not be seeing any inequalities of the kind:

$$\|u(t)\|_{L^2}^2 \lesssim \|u(0)\|_{L^2}^2.$$

Just to recap: the basic idea is to take advantage of the large numbers of conserved quantities by finding one for which the equilibrium is a local minimizer and satisfies some sort of convexity property. In some cases this is easier said than done, but in other cases, like the above, its not so bad. Much more general results are possible, in particular, one can consider more general equillibria and also more general stability criteria even for shear flows; see [HMRW85] and the references therein. Moreover, the general idea and variations thereof applies to a very wide variety of applications throughout plasma physics, atmospheric dynamics, and galaxy dynamics, to name a few.

As a last comment, Theorem 3.1 already tells us a lot, but on the other hand, it doesn't tell us about the behavior of higher norms. In this sense it doesn't settle certain questions about the actual dynamics of the solution: do solutions oscillate around in periodic or quasi-periodic orbits? do solutions develop all kinds of crazy small scales rapidly and become increasingly turbulent at the small scales? Do solutions settle back to shear flows in one way or another? The question of the long-time dynamics is in general very poorly understood – and here we do not mean we just cannot prove things, it is poorly understood even in the way *physicists* mean "understand". Naturally, it is this question we will be focusing on for the remainder of the course.

4 Mixing and dissipation in passive scalar flows at high Péclet number

Previously, we were mainly concerned with deducing spectral stability or nonlinear stability for planar shear flows. However, we neglected entirely the question of what the actual dynamics look like, which are far more interesting than the previous discussion makes it sound. One of the main dynamics we neglected to discuss was *mixing*. Here we will begin the discussion of these kinds of dynamics by first focusing on passive scalar flows, rather than the linearized fluid equations.

The mixing and dissipation of passive scalars in a given incompressible velocity field is given by the linear equation

$$\partial_t f + u \cdot \nabla f = \kappa \Delta f \tag{4.1a}$$
$$f(0) = f_{in}. \tag{4.1b}$$

for a scalar f, a given velocity field $u(t,x)$ with $\nabla \cdot u = 0$ and a diffusivity $\kappa > 0$. In these lectures we will not be concerned with regularity and well-posedness issues (not in the traditional sense anyway) so it will suffice to assume $u \in C^\infty_{t,x}$ and $f_{in} \in C^\infty_{t,x}$.

The advection diffusion equation is a classical and very important problem of practical and theoretical interest and certainly deserves to be studied in its own right. The hope is then that understanding certain aspects of this problem will also tell us something about hydrodynamic stability.

To non-dimensionalize we can replace (since the equation is linear we don't really need to rescale f)

$$f^\star(t,x) = f\left(\frac{t}{LU^{-1}}, \frac{x}{L}\right)$$
$$u^\star(t,x) = \frac{1}{U} u\left(\frac{t}{LU^{-1}}, \frac{x}{L}\right)$$

and we have

$$\partial_t f^\star + u^\star \cdot \nabla f^\star = \frac{\kappa}{LU}\Delta f.$$

The dimensionless number in front of the Δ is called the (inverse) *Péclet number*,

$$Pe = \frac{UL}{\kappa}.$$

It is a ratio of the time-scale of advective transport to the diffusive transport.

4.1 Passive scalar in Couette flow

Let us begin with the simplest of all examples: the planar Couette flow (dropping the $\star$'s and using $\kappa = Pe^{-1}$ as the inverse Peclet number)

$$\partial_t f + y\partial_x f = \kappa \Delta f \tag{4.2a}$$
$$f(0) = f_{in}. \tag{4.2b}$$

We will take periodic boundary conditions in x and infinite in y, so we have our problem on a cylinder $(x,y) \in \mathbb{T} \times \mathbb{R}$ (we could also consider in 3D or higher but nothing is different for passive scalars). The problem (4.2) was first solved by Lord Kelvin in 1887 [Kel87]. First, let us consider the case $\kappa = 0$. In this case, the solution is just

$$f(t, x, y) = f_{in}(x - ty, y).$$

Taking the Fourier transform gives the following:

$$\begin{aligned}
\hat{f}(t, k, \eta) &= \frac{1}{2\pi} \int e^{-ikx - i\eta y} f_{in}(x - ty, y) dx dy \\
&= \frac{1}{2\pi} \int e^{-ikx - ikty - i\eta y} f_{in}(x, y) dx dy \\
&= \hat{f}_{in}(k, \eta + kt).
\end{aligned}$$

For each k this is a linear-in-time transfer of information to high frequencies. This implies the lack of compactness in L^2 and the weak convergence back to equilibrium:

Exercise 4.1. For $f_{in} \in L^2$, prove that if f solves (4.2) with $f(0) = f_{in}$, then $f(t) \rightharpoonup \langle f_{in} \rangle_x$ in L^2. Show that this convergence is only strong if $f(t) = \langle f_{in} \rangle_x$ for all t. Similarly, prove that if $f_{in} \neq \langle f_{in} \rangle_x$ and $f_{in} \in H^n$, then $\| f(t) \|_{H^n} \approx \langle t \rangle^n \| f_{in} \|_{H^n}$ (we are denoting $\langle t \rangle = (1 + |t|^2)^{1/2}$).

The phenomenon of weak convergence despite the fact that we are on a compact set (sort of) is usually called *mixing*. Draw a picture or two of the Couette flow evolution to convince yourself that the linear evolution is not so dissimilar from some of the fundamental processes that take place when you stir milk into coffee.

Notice that the behavior in Exercise 4.1 is not possible in finite dimensional Hamiltonian systems. For the rest of the lectures, we will denote

$$\begin{aligned}
|k, \eta| &= |k| + |\eta| \\
\langle k, \eta \rangle &= (1 + |k, \eta|^2)^{1/2}.
\end{aligned}$$

We get the decay in negative Sobolev norms, at a price. For all $s \geq 0$,

$$\begin{aligned}
\| f - \langle f \rangle_x \|_{H^{-s}} &\lesssim \sum_{k \neq 0} \int \frac{1}{\langle k, \eta \rangle^{2s}} \left| \hat{f}_{in}(k, \eta + kt) \right|^2 d\eta \\
&\lesssim \sum_{k \neq 0} \int \frac{1}{\langle k, \eta \rangle^{2s} \langle \eta + kt \rangle^{2s}} \left| \langle \eta + kt \rangle^{2s} \hat{f}_{in}(k, \eta + kt) \right|^2 d\eta \\
&\lesssim \frac{1}{\langle t \rangle^s} \| f_{in} - \langle f_{in} \rangle_x \|_{H^s}.
\end{aligned}$$

One can view this as a more quantitative estimate on the weak convergence. We will re-visit the loss of regularity in this formula at length later. Notice that if we take the Fourier transform of $\partial_t f + y \partial_x f = 0$ we get

$$\partial_t \hat{f} - k \partial_\eta \hat{f} = 0,$$

which is still a shear flow. This is an important point: mixing in Couette flow is transport to infinity *in frequency.*

Consider now the diffusive case, for $\kappa > 0$

$$\partial_t f + y \partial_x f = \kappa \Delta f.$$

In this case, Lord Kelvin, in [Kel87], defined the variables $X = x - ty$ and $g(t, X, y) = f(t, X + ty, y)$, which then solves

$$\partial_t g = \kappa \Delta_L g$$
$$\Delta_L = \partial_{XX} + (\partial_y - t\partial_X)^2.$$

The 'L' stands for 'linear' for reasons which will make more sense later. Taking the Fourier transform and then integrating gives

$$\partial_t \hat{g} = -\kappa \left(k^2 + |\eta - kt|^2 \right) \hat{g},$$

and

$$\hat{g}(t, k, \eta) = \hat{f}_{in}(k, \eta) \exp \left[-\kappa \int_0^t \left(k^2 + |\eta - k\tau|^2 \right) d\tau \right]. \tag{4.3}$$

Notice that we have the following bound

$$\int_0^t |\eta - k\tau|^2 \, d\tau \gtrsim \min(|\eta|^2 \, t, k^2 t^3).$$

To see this, consider separately contributions to the integral from $\tau \leq \frac{\eta}{2k}$ and $\tau \geq 2\frac{\eta}{k}$. From (4.3), this shows that we get the following *enhanced dissipation* estimate for some c (which happens to be $< 1/3$),

$$\|g_{\neq}\|_{L^2} \lesssim \|g_{in}\|_{L^2} e^{-c\nu t^3}$$

The key point to the decay is the relationship between ν and t. The characteristic time-scale when the dissipation begins to dominate is $\tau_{ED} \sim \nu^{-1/3}$, which is significantly faster than the ν^{-1} time-scale associated with the heat equation. The Couette flow is sending information to high frequencies linearly in time, and this is where the 3 comes from (order of the Laplacian in the damping plus one from the integral). One can imagine this relaxation mechanism like the way a cup of coffee relaxes after you stir it up into a vortex. First, the angular dependence is eliminated as the fluid stirring itself mixes information to high frequencies where it is rapidly dissipated, like the relaxation of $k \neq 0$ modes. Over a longer time scale the (approximately) radially symmetric mean vortex relaxes in place as a laminar flow.

4.2 More general shear flows

The Couette flow is so easy we might get the impression that more general problems will continue to be super easy. This is not correct, and surprisingly little is known in mathematical rigor about more general flows (which is not to say that nothing is known or that there exists no good work on this – that is far from the truth, see e.g. [CKRZ, BW13, GGN09, Den13, BCZGH15, Zil14a, Zil14b] to name a small subset of related works, however our knowledge is still quite limited relative to what might be desired).

Following how we approached the Couette flow, the first goal is to consider the inviscid problem and study the transfer of information to high frequencies. Getting enhanced dissipation rates is in general harder, though see [CKRZ, BW13] for some information on this. Let us first try with a humble goal of considering more general shear flows than Couette; in this case results are not difficult.

We will prove the following two basic theorems, the proof is in a style similar to some found in [Zil14a] combined with some ideas from the method of stationary phase for classical oscillatory integrals [Ste93]. I will state a version on $\mathbb{T} \times \mathbb{R}$ and a version on $\mathbb{T}^2$, but it admits suitable generalizations to more general tori, channels bounded by walls with no flux boundaries, and higher dimensions.

Theorem 4.1 (Mixing by shear flows in $\mathbb{T} \times \mathbb{R}$). *Let $U(y) \in C^\infty$ and let f solve the PDE*

$$\partial_t f + U(y)\partial_x f = 0.$$

Then

(i) If there is some $\delta > 0$ such that $|U'(y)| \geq \delta$ for all y, then

$$\|f(t) - \langle f\rangle_x\|_{H^{-1}} \lesssim \langle t\rangle^{-1}\|f(0)\|_{L^2_x H^1_y}. \tag{4.4}$$

(ii) If there is some $\delta > 0$ such that $|U''(y)| \geq \delta$ for all y, then

$$\|f(t) - \langle f\rangle_x\|_{H^{-1}} \lesssim \langle t\rangle^{-1/2}\|f(0)\|_{L^2_x H^1_y}. \tag{4.5}$$

(iii) More generally, suppose there is some R and δ such that $|U'(y)| \geq \delta$ for all $|y| \geq R$ and further that there are finitely many points y_i, $1 \leq i \leq K$, such that $U'(y_i) = 0$ and finitely many inflection points where $U''(\tilde{y}_i) = 0$. Further suppose that U' degenerates only to finite order: that is, there is a finite $n \in \mathbb{N}$ which is the minimal integer such that $U^{(n)}(y_i) \neq 0$ for all y_i (the critical points of the flow). Then,

$$\|f(t) - \langle f\rangle_x\|_{H^{-1}} \lesssim \langle t\rangle^{-1/n}\|f(0)\|_{L^2_x H^1_y}. \tag{4.6}$$

Remark 8. Items (i) and (ii) are special cases of item (iii), however, the statements are less technical and the proofs can be made much more direct in the cases of (i) and (ii).

This theorem has the following analogue on $\mathbb{T}^2$, which we only state in the general case now.

Theorem 4.2 (Mixing by shear flows in $\mathbb{T}^2$). *Let $U(y)$ be C^∞ with finitely many points y_i, $1 \leq i \leq K$, such that $U'(y_i) = 0$ and finitely many inflection points where $U''(\tilde{y}_i) = 0$. Suppose that U' degenerates only to finite order: that is, there is a finite $n \in \mathbb{N}$ which is the minimal integer such that $U^{(n)}(y_i) \neq 0$ for all y_i (the critical points of the flow); necessarily $n \geq 2$. Let f solve the PDE*

$$\partial_t f + U(y)\partial_x f = 0.$$

Then, we have the decay rate:

$$\|f(t) - \langle f\rangle_x\|_{H^{-1}} \lesssim \langle t\rangle^{-1/n}\|f(0)\|_{L^2_x H^1_y}. \tag{4.7}$$

Remark 9. Notice the loss of regularity in (4.7). From the example of the Couette flow, we can surmise that this is necessary to deduce the pointwise-in-time decay estimate.

Remark 10. On $\mathbb{T}^2$ there are always critical points, and hence this theorem shows that the H^{-1} convergence rate is always at most $\langle t\rangle^{-1/2}$.

Remark 11. The proof will show that one can be a tiny bit more precise about exactly the norm that appears on the RHS of (4.7).

Proof. We may assume without loss of generality $\langle f \rangle_x = 0$ and $t \geq 1$. First observe that

$$\|f\|_{H^{-1}} = \sup_{\phi \in H^1 : \|\phi\|_{H^1} = 1; \phi} \int f \phi dA.$$

Let ϕ be such an arbitrary test function. Fourier transform in x only, denoting this as $\hat{f}_k(t, y)$, we get

$$\begin{aligned}\left| \int f \phi dA \right| &= \left| \sum_k \int \hat{f}_k(0, y) e^{ikU(y)t} \overline{\hat{\phi}}_k(y) dy \right| \\ &\leq \sum_k \left| \int \hat{f}_k(0, y) e^{ikU(y)t} \overline{\hat{\phi}}_k(y) dy \right|.\end{aligned}$$

The result will follow by the method of stationary phase (see e.g. [Ste93]). Indeed, let $\chi_\epsilon(y)$ be a smooth cutoff function supported in ϵ intervals around the critical points y_i. Then consider the two contributions separately:

$$\begin{aligned}\left| \int \hat{f}_k(0, y) e^{ikU(y)t} \overline{\hat{\phi}}_k(y) dy \right| &\leq \left| \int (1 - \chi_\epsilon(y)) \hat{f}_k(0, y) e^{ikU(y)t} \overline{\hat{\phi}}_k(y) dy \right| + \left| \int \chi_\epsilon(y) \hat{f}_k(0, y) e^{ikU(y)t} \overline{\hat{\phi}}_k(y) dy \right| \\ &= T1 + T2.\end{aligned}$$

On $T2$, the phase $ikU(y)$ is stationary so we cannot use any integration by parts. However, we can employ (together with the $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ embedding):

$$T2 \lesssim \|\chi_\epsilon\|_{L^1} \sum_k \|f_k\|_{L^\infty_y} \|\phi_{-k}\|_{L^\infty_y} \lesssim \epsilon \|f\|_{L^2_x H^1_y}.$$

For $T1$ we have to be a little more precise. On $T1$ we may employ $\frac{1}{ikU'(y)t} \frac{d}{dy} e^{ikU(y)t} = e^{ikU(y)t}$ and integrate by parts to deduce

$$\begin{aligned}T1 &\lesssim \frac{1}{\langle t \rangle} \sup_y \frac{\mathbf{1}_{|y-y_i|>\epsilon}}{|U'(y)|} \sum_k \frac{1}{|k|} \left(\|\hat{f}_k\|_{L^2_y} \|\nabla \phi_{-k}\|_{L^2_y} + \|\nabla \hat{f}_k\|_{L^2_y} \|\phi_{-k}\|_{L^2_y} \right) \\ &+ \sum_{k \neq 0} \frac{1}{t\,|k|} \left| \int e^{ikU(y)t} \frac{1}{U'(y)} \partial_y \chi_\epsilon(y) \hat{f}(k, y) \overline{\hat{\phi}}(k, y) dy \right| \\ &+ \frac{1}{t\,|k|} \left| \int e^{ikU(y)t} \left(\frac{d}{dy} \frac{1}{U'(y)} \right) (1 - \chi_\epsilon(y)) \hat{f}(k, y) \overline{\hat{\phi}}(k, y) dy \right| \\ &= T1_0 + T1_1 + T1_2.\end{aligned}$$

The treatment of $T1_1$ is straightforward

$$\begin{aligned}T1_1 &\lesssim \frac{1}{\langle t \rangle} \sup_y \frac{\mathbf{1}_{|y-y_i|>\epsilon}}{|U'(y)|} \sum_{k \neq 0} \|\partial_y \chi_\epsilon\|_{L^1} \|\hat{f}_k\|_{H^1} \|\hat{\phi}_k\|_{H^1}. \\ &\lesssim \frac{1}{\langle t \rangle} \sup_y \frac{\mathbf{1}_{|y-y_i|>\epsilon}}{|U'(y)|} \|\hat{f}_k\|_{L^2 H^1} \|\hat{\phi}_k\|_{L^2 H^1}.\end{aligned}$$

By non-degeneracy (and Taylor's theorem) we get

$$\sup_y \frac{|1 - \chi_\epsilon(y)|}{|U'(y)|} \approx \sup_y \frac{\mathbf{1}_{|y-y_i|>\epsilon}}{|U'(y)|} \lesssim \epsilon^{-n+1}.$$

For $T1_2$ we have to be a bit more precise due to the derivative of the nearly singular $(U')^{-1}$. Let $\tilde{\chi}_\epsilon(y)$ be a smooth cutoff supported in ϵ^{n-1} intervals around the *inflection* points. Then we further divide

$$\begin{aligned}
T1_2 &\leq \frac{1}{\langle t\rangle}\left|\int (1-\chi_\epsilon(y))\,(1-\tilde{\chi}_\epsilon(y))\left(\frac{d}{dy}\frac{1}{U'(y)}\right)\hat{f}_k(0,y)e^{ikU(y)t}\overline{\hat{\phi}}_k(y)dy\right| \\
&\quad+\frac{1}{\langle t\rangle}\left|\int (1-\chi_\epsilon(y))\,\tilde{\chi}_\epsilon(y)\left(\frac{d}{dy}\frac{1}{U'(y)}\right)\hat{f}_k(0,y)e^{ikU(y)t}\overline{\hat{\phi}}_k(y)dy\right| \\
&= T1_{21}+T1_{22}.
\end{aligned}$$

For $T1_{21}$ we note that the U' is monotone on each disjoint interval where the integral is supported (call these intervals J_i, $1\leq i\leq N$) and U' is bounded below by $|U'|\gtrsim \epsilon^{1-n}$ on the support of the integrand. Therefore (using again the embedding $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$),

$$\begin{aligned}
T1_{21} &\lesssim \frac{1}{\langle t\rangle}\|f\|_{L^2L^\infty}\|\phi\|_{L^2L^\infty}\sum_{i=1}^{N}\int_{J_i}\left|\frac{d}{dy}\frac{1}{U'(y)}\right|dy \\
&= \frac{1}{\langle t\rangle}\|f\|_{L^2H^1}\|\phi\|_{L^2H^1}\sum_{i=1}^{N}\left|\int_{J_i}\frac{d}{dy}\frac{1}{U'(y)}dy\right| \\
&\leq \frac{2N}{\langle t\rangle}\|f\|_{L^2H^1}\|\phi\|_{L^2H^1}\sup_{y\in J_i}\frac{1}{|U'(y)|} \\
&\lesssim \frac{1}{\langle t\rangle\epsilon^{n-1}}\|f\|_{L^2H^1}\|\phi\|_{L^2H^1}.
\end{aligned}$$

For $T1_{22}$ we use instead that $|\tilde{\chi}_\epsilon U''|\lesssim \epsilon^{n-1}$ because U is smooth. From this we have

$$\begin{aligned}
T1_{22} &\lesssim \frac{1}{\langle t\rangle}\left|\int (1-\chi_\epsilon(y))\,\tilde{\chi}_\epsilon(y)\left(\frac{U''(y)}{(U'(y))^2}\right)\hat{f}_k(0,y)e^{ikU(y)t}\overline{\hat{\phi}}_k(y)dy\right| \\
&\lesssim \frac{1}{\langle t\rangle}\|\tilde{\chi}_\epsilon U''\|_{L^\infty_y}\epsilon^{2-2n}\|f\|_{L^2H^1}\|\phi\|_{L^2H^1} \\
&\lesssim \frac{1}{\langle t\rangle}\epsilon^{1-n}\|f\|_{L^2H^1}\|\phi\|_{L^2H^1}.
\end{aligned}$$

Putting everything together we get

$$\|f(t)\|_{H^{-1}}\lesssim T1+T2\lesssim\left(\epsilon+\frac{1}{\langle t\rangle\epsilon^{n-1}}\right)\|f(0)\|_{L^2H^1}.$$

Hence, we can choose the optimal ϵ,

$$\epsilon\approx\langle t\rangle^{-\frac{1}{n}}.$$

and deduce

$$\|f(t)\|_{H^{-1}}\lesssim\langle t\rangle^{-\frac{1}{n}}\|f(0)\|_{L^2_xH^1_y},$$

which completes the proof. □