DAY 1

Regular vs. Singular perturbations & boundary layer correctors

Note Title 8/2/2015

I Regular perturbations

- Consider the linear problem:

$$A u - \varepsilon B u = f$$

where $\varepsilon > 0$ is a parameter, f is a fixed RHS, and where A & B are linear operators $X \to Y$, such that A is "dominant": $\to A : X \to Y$ invertible

$\to A^{-1}B : X \to X$ is a bounded operator.

- Then the solution $u = u^\varepsilon$ of the above problem is given by the inviscid solution $\bar{u} + O(\varepsilon)$.

More precisely, we may write (this is called an ansatz)

$$u = \sum_{n \geq 0} \varepsilon^n u_n \quad ; \text{ where } u_0 = \bar{u}$$

and one may compute u_n iteratively by:

$$\sum_{n \geq 0} \varepsilon^n A u_n - \sum_{n \geq 0} \varepsilon^{n+1} B u_n = f$$

$$\Longleftrightarrow \quad \sum_{m \geq 1} \varepsilon^m \left(A u_m + B u_{m-1} \right) = 0$$

using $u_0 = \bar{u} = A^{-1} f$

$$\Rightarrow \text{ set } \begin{cases} u_m = A^{-1} B \, u_{m-1} & \forall m \geq 1 \\ u_0 = A^{-1} f \end{cases}$$

- if $A^{-1}B$ is bounded, say by M

$$\| \varepsilon^m u_m \|_X \leq \varepsilon^m M \, \| u_{m-1} \|_X \leq \varepsilon^m M^m \, \|A^{-1} f\|_X$$

so that the formal power series converges to a function $u \in X$ $\forall$ $\varepsilon \in (0, M)$.

- Moreover, $\| u - \bar{u} \|_X \lesssim \varepsilon \to 0$ as $\varepsilon \to 0$

- An alternative way to reach the same conclusion is:

$$(1 - \varepsilon A^{-1} B) u = A^{-1} f$$

$$\Rightarrow \quad u = \frac{1}{1 - \varepsilon A^{-1} B} \, (A^{-1} f)$$

$$= \sum_{n \geq 0} (\varepsilon A^{-1} B)^n \, (A^{-1} f)$$

$$= \sum_{n \geq 0} \varepsilon^n \underbrace{(A^{-1} B)^n (A^{-1} f)}_{u_n}$$

$\Rightarrow$ <u>note</u>: $n = 0$ yields $A^{-1} f = u_0 = \bar{u}$.

<u>Example</u>: $-\Delta u + \varepsilon a \partial_x u = f$

on $\mathbb{T}^1$; with periodic & <u>zero mean</u> assumption (also on f?)

the operator $\widehat{(-\Delta)^{-1} a \partial_x}(k) = \dfrac{iak}{|k|^2}$

is given by a multiplier bdd. by a $\forall k \in \mathbb{Z} \setminus \{0\}$ and thus the solution is a regular expansion in ε of the (!) solution of the limiting problem, $\bar{u} = (-\Delta)^{-1} f$.

II Singular Perturbations

- **Example 1: First order ODE.**

  - Consider on $\mathbb{R}_+ = [0,\infty)$ the ordinary differential equation:

$$\begin{cases} \varepsilon u_x + u = 1 \\ u\big|_{x=0} = u_0 \neq 1 \end{cases}$$

- The solution of the limiting equation ($\varepsilon = 0$):

$$\bar{u} = 1$$

  is unique, but doesn't obey the B.C. for the problem with $\varepsilon > 0$, as $u_0 \neq 1$.

- We call this a mismatch in boundary conditions.

  - $\varepsilon > 0$: we can solve the ODE explicitly by using integrating factors:

$$u_x + \frac{1}{\varepsilon} u = \frac{1}{\varepsilon}$$

$$\Rightarrow \quad \partial_x \left(u e^{x/\varepsilon} \right) = \frac{1}{\varepsilon} e^{x/\varepsilon}$$

$$\Rightarrow \quad u(x) e^{x/\varepsilon} - u(0) = \frac{1}{\varepsilon} \int_0^x e^{y/\varepsilon} \, dy$$

$$\Rightarrow \quad u(x) = u_0 e^{-x/\varepsilon} + e^{-x/\varepsilon} \left(e^{x/\varepsilon} - 1 \right)$$

$$= (u_0 - 1) e^{-x/\varepsilon} + \underbrace{1}_{\hookrightarrow \ \bar{u}\text{, the sol}^{n} \text{ for } \varepsilon = 0}$$

- Thus the solution for $\varepsilon > 0$ is given by the solution for $\varepsilon = 0$, when $\frac{x}{\varepsilon} \gg 1$ (as $\varepsilon \to 0$) and is given by a boundary layer corrector

$$u_{BL}(x) = (u_0 - 1)\, e^{-x/\varepsilon}$$

for all $0 < \boxed{\frac{x}{\varepsilon}} \lesssim 1$. → we call this the characteristic length scale of the corrector

- Note: the boundary condition for the corrector is

$$u_{BL}(0) = u_0 - 1 = u(0) - \bar{u}(0)$$

and the corrector in fact only depends on a "fast variable" $y = x/\varepsilon$:

$$u_{BL}(y) = (u_0 - 1)\, e^{-y}.$$

- what happens in this case in the singular limit $\varepsilon \to 0$?

In L^∞: $\sup_{x \geq 0} |u(x) - \bar{u}(x)|$

$$= \sup_{x \geq 0} \left| (u_0 - 1)\, e^{-x/\varepsilon} \right| = |u_0 - 1| \neq 0$$

thus: $\lim_{\varepsilon \to 0} \| u - \bar{u} \|_{L^\infty} \neq 0$!

In L^2: $$\| u - \bar{u} \|_{L^2(\mathbb{R}_+)} = |u_0 - 1| \left(\int_0^\infty e^{-2x/\varepsilon}\, dx \right)^{1/2}$$

$$= |u_0 - 1| \frac{\sqrt{\varepsilon}}{\sqrt{2}}$$

thus: $\lim_{\varepsilon \to 0} \| u - \bar{u} \|_{L^2} = 0$ (at rate $\sqrt{\varepsilon}$).

- The singular limit is thus sensitive to the topology.

- Physically it makes sense (and this will be useful for a more general class of equations) to

→ Make an ansatz: $\boxed{u(x) = \bar{u}(x) + u_{BL}(x/\delta) + O(\delta)}$

→ Formally derive what δ should equal, in terms of ε, and formally derive an equation for u_{BL}

→ Hope that the conjectured $O(\delta)$ term is trully $o(\delta)$... this is what physicists usually never do... and thus their predictions for boundary layer behaviour are FORMAL. (though accurate usually :))

- Inserting the ansatz $u(x) = \bar{u}(x) + u_{BL}(x/\delta)$

where $u_{BL}(0) = u(0) - \bar{u}(0) = u_0 - 1$

$u_{BL}(\infty) = 0$ → the $\bar{u}$ flow should be ok from the boundary

we arrive at:

$$\frac{\varepsilon}{\delta} u_{BL}''(y) + u_{BL}'(y) = 0 \qquad \text{where } y = x/\delta$$

⇒ postulate $\boxed{\varepsilon = \delta}$ and solve

AS BEFORE

$$\left.\begin{array}{l} u_{BL}' + u_{BL} = 0 \\ u_{BL}\big|_{x=0} = u_0 - 1 \end{array}\right| \Rightarrow u_{BL} = (u_0 - 1)\, e^{-y} = (u_0 - 1)\, e^{-x/\delta}$$

- **Note** in the above example there was no $O(\delta)$ term, i.e. $\bar{u} + u_{BL}$ was an **exact** solution.

Example 2: Second order ODE:

- Fix $a \neq 0$; $f(x) \in C^\infty \cap L^1 \cap L^\infty$, and consider on $\mathbb{R}_+$:

$$\begin{cases} -\varepsilon \partial_x^2 u + a \partial_x u + u = f(x) & \text{on } \mathbb{R}_+ \\ u\big|_{x=0} = 0 & \text{on } \partial\mathbb{R}_+ \end{cases}$$

also $u\big|_{x=\infty} = 0$; inherent

- This is a 2^{nd} order ODE, with Dirichlet B.C., and it may be shown to have solutions

→ E.g. by Lax-Milgram:

- The weak form of the ODE is $B(u,v) = 0$

where $B(u,v) = \int_{\mathbb{R}_+} \left(\varepsilon \partial_x u \, \partial_x v - a u \partial_x v + uv \right) dx$

- $B : H_0^1 \times H_0^1 \to \mathbb{R}$ $\quad \left(H_0^1(\mathbb{R}_+) = \{ f \in H^1(\mathbb{R}_+) \text{ s.t. } f|_{x=0} = f|_{x=\infty} = 0 \} \right)$

is → bilinear

→ bounded, i.e.

$$|B(u,v)| \leq \varepsilon \|u\|_{\dot{H}^1} \|v\|_{\dot{H}^1} + a \|u\|_{L^2} \|v\|_{\dot{H}^1} + \|u\|_{L^2} \|v\|_{L^2}$$
$$\leq (\varepsilon + a + 1) \|v\|_{H^1} \|u\|_{H^1} \qquad \forall\, u, v \in H_0^1$$

→ coercive:

$$B(u,u) = \varepsilon \|u\|_{\dot{H}^1}^2 + \|u\|_{L^2}^2 \geq \min\{\varepsilon, 1\} \|u\|_{H^1}^2 \qquad \forall\, u \in H_0^1$$

- Thus, given $f \in L^2$, we may define the BOUNDED linear functional

$$F(v) = \langle f, v \rangle = \int v f \, dx \quad \in (H_0^1)^*$$

- The Lax-Milgram theorem guarantees that

$$\exists! \; u \in H_0^1 \quad \text{s.t.} \quad B(u,v) = F(v) \quad \forall v \in H_0^1$$

↳ proof uses Riesz-Representation Theorem ⊕ Contraction mapping if B not symmetric

- As before, let's solve the limiting "inviscid" equation, and see if it obeys the correct boundary conditions.

- $\underline{\varepsilon = 0}$ $\quad a\,\partial_x \bar{u} + \bar{u} = f \quad \& \quad u|_{x=\infty} = 0$ (inviscid equation requires only one boundary condition)

$$\Rightarrow \partial_x\left(\bar{u}\, e^{x/a}\right) = \frac{1}{a} f(x)\, e^{x/a}$$

- when $a > 0$: $\quad \Rightarrow \bar{u}(x) = \frac{1}{a} \int_0^x e^{-\frac{(x-y)}{a}} f(y)\, dy$

 ($\underline{\text{note}}$: if $f \in L^1\left(e^{y/a} dy\right)$ this obeys $u(x) \to 0$ as $x \to \infty$)

 moreover, this solution obeys $\bar{u}|_{x=0} = 0$ and thus, the boundary conditions match, and we may expect the viscous solution to be a regular perturbation of $\bar{u}(x)$.

- when $a < 0$; integrating on (x, ∞) we get:

$$\bar{u}(x) = \frac{1}{(-a)} \int_x^\infty e^{-\frac{y-x}{(-a)}} f(y)\, dy$$

 (the B.C. as $x \to \infty$ is automatically obeyed)

- However the above solution obeys:

$$\bar{u}(0) = \frac{1}{(-a)} \int_0^\infty e^{-\frac{y}{(-a)}} f(y)\, dy \underbrace{\neq 0}_{\text{if e.g. } \boxed{f \geq 0}}$$

- Thus in this case we have a mismatch in the boundary conditions, and a BL corrector is needed.

- $\underline{\varepsilon > 0}$, $\underline{a > 0}$:

  Define $L = (a\partial_x + 1)^{-1}$ by:

$$Lg(x) = \frac{1}{a} \int_0^x e^{-\frac{x-y}{a}} g(y)\, dy$$

  we have

$$\begin{cases} a\partial_x(Lg) + Lg = g \\ Lg\big|_{x=0} = 0 \quad ; \quad \underbrace{Lg\big|_{x=\infty} = 0 \quad \text{if } g \to 0 \text{ as } x \to \infty}_{\left(\begin{array}{l}\text{because } L : L^1 \to L^1 \\ \text{and we have } H^1_0 \Rightarrow C^{1/4} \text{ functions}\end{array}\right)} \end{cases}$$

- Then we may write $u = \bar{u} + \sum_{n \geq 1} \varepsilon^n u_n$

  and obtain that

$$u_1 = L(\bar{u}_{xx})$$
$$u_2 = L(-u_{1xx})$$
$$\vdots$$

  and thus $u = \bar{u} + O(\varepsilon)$ even in L^∞ !

- $\underline{\varepsilon > 0;\ a < 0:}$

$$\bar{u} = \frac{1}{(-a)} \int_x^\infty e^{-\frac{y-x}{(-a)}} f(y)\, dy$$

doesn't obey the boundary conditions, a mismatch.

- make the ansatz that:

$$u(x) = \bar{u}(x) + u_{BL}(x/\delta) + O(\delta)$$

with
$$\begin{cases} u_{BL}(0) = -\bar{u}(0) = \frac{1}{a} \int_0^\infty e^{-\frac{y}{(-a)}} f(y)\, dy \\ u_{BL}(\infty) = 0 \end{cases}$$

inserting the ansatz in the equation and letting $y = x/\delta$, we obtain:

$$-\frac{\varepsilon}{\delta^2} u''_{BL}(y) + \frac{a}{\delta} u'_{BL}(y) + u_{BL}(y) = \varepsilon\, \partial_{xx} \bar{u}(x)$$

- we next choose $\delta = \varepsilon$ so that the first two terms are balanced:

$$\frac{1}{\varepsilon}\left(-u''_{BL}(y) + a\, u'_{BL}(y)\right) + u_{BL}(y) = \varepsilon\, \partial_{xx} \bar{u}(x)$$

and we impose that:

$$\begin{cases} -u''_{BL}(y) + a\, u'_{BL}(y) = 0 \\ u_{BL}(0) = -\bar{u}(0) \end{cases}$$

$$\Rightarrow \quad u_{BL}(y) = -\bar{u}(0)\, e^{ay} = -\bar{u}(0)\, e^{-(-a)y} \underset{\text{as } y \to \infty}{\to} 0$$

Thus, inserting this back in the equation, we see that the error made is:

$$(-\varepsilon \partial_{xx} + a\partial_x + 1)\left(\bar{u}(x) + u_{BL}(x/\varepsilon)\right)$$

$$= \underbrace{\bar{u}(0)\, e^{-(-a)x/\varepsilon}}_{\substack{= O(\varepsilon) \text{ in } L^1 \\ = O(\sqrt{\varepsilon}) \text{ in } L^2 \\ = O(1) \text{ in } L^\infty\,!}} + \underbrace{\varepsilon\, \partial_{xx} \bar{u}(x)}_{\substack{O(\varepsilon) \text{ in } L^1 \\ O(\varepsilon) \text{ in } L^2}}$$

- Thus, although we have fixed the boundary condition, we can only obtain that

$$\| u - \bar{u} - u_{BL} \|_{L^p} \to 0 \quad \text{as } \varepsilon \to 0 \text{ for } p<\infty$$

and it seems clear that a higher order expansion in ε is needed as an ansatz.

III Is NSE → Euler sometimes a good limit?

→ In general $\bar{u}|_{\partial\Omega}$ may not be 0, (denoting $\bar{u} = u^E$) while always $u|_{\partial\Omega} = 0$ (denoting $u = u^{NS}$) and thus we may have a mismatch of boundary conditions.

→ However there are solutions of Euler which do obey $\bar{u}|_{\partial\Omega} = 0$, and the simplest ones we can think of are stationary solutions.

→ <u>Example</u>: consider two dimensions, and fix $R>0$.

Assume $\bar{\omega}_0$ is radial, and supported in $B_R(0)$, with 0-mass, i.e. $\int_{B_R} \bar{\omega}_0(x)\,dx = 2\pi \int_0^R r\,\bar{\omega}_0(r)\,dr = 0$

Then, from 2D Biot-Savart, we obtain

$$\bar{u}_0(x) = \frac{x^\perp}{|x|^2} \int_0^{|x|} r\,\bar{\omega}_0(r)\,dr$$

and thus:
$$\begin{cases} \bar{u}_0 \cdot \nabla \bar{\omega}_0 = 0 & \text{since } x^\perp \cdot x = 0 \\ \bar{u}_0 \big|_{\partial B_R} = 0 & \end{cases}$$

This is thus a stationary solution of 2D Euler, with 0, boundary conditions for the velocity.

→ What happens for 2D NSE with this initial datum?

$\left(\Delta \text{ in polar } \partial_r^2 + \frac{1}{r}\partial_r \right)$

- The vorticity stays <u>radial</u>, and obeys the heat equation:

$$\partial_t \omega - \nu \Delta \omega = \left[\partial_t - \nu \left(\partial_r^2 + \frac{1}{r}\partial_r \right) \right] \omega(r,t) = 0$$

→ the nonlinearity vanishes, since $\begin{pmatrix} u(x) \sim x^\perp \\ \nabla\omega(x) \sim x \end{pmatrix}$.

- The explicit solution is:

$$\omega(x,t) = e^{\nu t\Delta}\omega_0(x)$$
$$= \frac{1}{4\pi\nu t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4\nu t}}\,\omega_0(|y|)\,dy$$

$$u(x,t) = \frac{x^\perp}{|x|^2}\int_0^{|x|} r\,\omega(r,t)\,dr.$$

- Note: $(\omega-\omega_0)(t) = \left(e^{\nu t\Delta} - Id\right)\omega_0$

- Thus, if $\omega_0 \in L^2$:

$$\|u(t)-\bar{u}\|_{L^2} = \|(-\Delta)^{-1/2}\left(e^{\nu t\Delta} - Id\right)\omega_0\|_{L^2}$$
$$\leq \|\omega_0\|_{L^2}\cdot \underbrace{\sup_\xi\left(\frac{1}{|\xi|}\left(e^{-\nu t|\xi|^2}-1\right)\right)}_{\lesssim \sqrt{\nu t}}$$

- If $\omega_0 \in H^1$:

$$\|u(t)-\bar{u}(t)\|_{L^2} \leq \|\omega_0\|_{H^1}\ \underbrace{\sup_\xi\left(\frac{1}{|\xi|^2}\left(e^{-\nu t|\xi|^2}-1\right)\right)}_{\leq \nu t}$$

- If $\omega_0 \in H^{-1}$: we get <u>no rate</u>.

- Thus, the <u>rate of vanishing depends on the smoothness</u> of the initial datum, even in the case of regular perturbations.

Inviscid limit without boundaries

- Say we are working on $\mathbb{R}^d$ (or $\mathbb{T}^d$), and the initial datum for Euler and NSE are smooth, and "close" (wrt ν). Then the inviscid limit holds in a smooth topology, with rates. More precisely, we have:

Theorem [Swann '71 ; Kato '72 ; Constantin '86 ; Masmoudi '06]

- Let $\bar{u}_0 \in H^{m+1}$; $m > \frac{d}{2}+1$, and let $\bar{u}$ be the smooth solution of d-dimensional Euler on $[0, T_0]$; where $T_0 = T_0(\|\bar{u}_0\|_{H^{m+1}}) > 0$.
- Let M_0 be such that $\|\bar{u}_0\|_{H^{m+1}} \leq M_0$, and let M_1 be such that $\int_0^{T_0} \|\bar{\omega}(t)\|_{L^\infty} dt \leq M_1$.
- Then there exists $\nu_0 = \nu_0(T_0, M_0, M_1) > 0$ such that $\forall \, \nu \in (0, \nu_0]$ we have:

$$\sup_{t \in [0, T_0]} \|u(t) - \bar{u}(t)\|_{H^m}^2 \leq \nu t \cdot \gamma_m$$

where $\gamma_m = \gamma_m(T_0, M_0, M_1, m)$; and u the solution of NSE, with datum u_0.

<u>Proof</u>: We let $v = u - \bar{u}$ be the difference btw. the Navier-Stokes and the Euler solution. It obeys the PDE

$$\begin{cases} \partial_t v + \bar{u} \cdot \nabla v + v \cdot \nabla v + v \cdot \nabla \bar{u} - \nu \Delta u + \nabla q = 0 \\ \nabla \cdot v = 0 \end{cases}$$

where q is the difference of the NSE & Euler pressures.

- In L^2 we have:

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = -\int v\cdot\nabla\bar{u}\cdot v\,dx - \nu\int \Delta u\cdot\bar{u}\,dx$$

$$\leq \|v\|_{L^2}^2\|\nabla\bar{u}\|_{L^\infty} + \frac{\nu}{2}\|\nabla u\|_{L^2}^2 + C\nu\|\nabla\bar{u}\|_{L^2}^2$$

- In $\dot{H}^m$ we have (assuming $m > d/2$)

$$\frac{1}{2}\frac{d}{dt}\|v\|_{\dot{H}^m}^2 + \nu\|\nabla u\|_{\dot{H}^m}^2$$

$$= -\nu\langle\Delta u, \bar{u}\rangle_{\dot{H}^m} - \langle v\cdot\nabla v, v\rangle_{\dot{H}^m} - \langle v\cdot\nabla\bar{u}, v\rangle_{\dot{H}^m} - \langle\bar{u}\cdot\nabla v, v\rangle_{\dot{H}^m}$$

$$\leq \frac{\nu}{2}\|\nabla u\|_{\dot{H}^m}^2 + C\nu\|\bar{u}\|_{\dot{H}^{m+1}}^2 + C\|\bar{u}\|_{H^{m+1}}\|v\|_{H^m}^2 + C\|v\|_{H^m}^3$$

- Thus, combining the two, we obtain that for $\nu > 0$ we have

$$\boxed{\frac{d}{dt}\|v\|_{H^m}^2 \leq C\nu\|\bar{u}\|_{H^{m+1}}^2 + C\|\bar{u}\|_{H^{m+1}}\|v\|_{L^2}^2 + C\|v\|_{H^m}^3}$$

- On $[0, T_0]$, we know that <u>independent of viscosity</u> we have:

$$\sup_{[0,T_0]}\|\bar{u}\|_{H^{m+1}} \leq K_0(T_0, M_0, M_1) =: K_0.$$

- Inserting the above, we arrive at:

$$\frac{d}{dt}\|v\|^2_{H^m} \leq C_0 k_0^2\, \nu \;+ C_0 k_0 \|v\|^2_{L^2} + C_0 \|v\|^3_{L^2}$$

where C_0 is a universal constant $\left(C_0 = C_0(d, m) \right)$.

- Recall that the initial datum is $v|_{t=0} = 0$!

- Let's estimate how long it takes so that $\|v\|_{H^m}$ reaches 1. Define $T_* = \sup_{t \geq 0} \{ t : \|v(t)\|_{H^m} \leq 1 \}$.

- Then on $[0, T_*]$ we in fact have: $\|v\|^3_{H^m} \leq \|v\|^2_{H^m}$ and thus

$$\frac{d}{dt}\|v\|^2_{H^m} \leq C_0 (k_0 + 1) \|v\|^2_{H^m} + \nu C_0 k_0^2$$

which yields, by the comparison principle for ODEs that on $[0, T_*]$:

$$\|v(t)\|^2_{H^m} \leq \nu t \left(C_0 k_0^2 \exp\left(C_0 (k_0+1) t \right) \right)$$
$$\leq \nu T_* C_0 k_0^2 \exp\left(C_0 (k_0+1) T_* \right).$$

- If we wish to take $T_* \geq T_0$, it follows that we just need that: $\nu \leq \nu_0$, where ν_0 is s.t.

$$\nu_0 = \frac{1}{T_0 \exp\left(C_0 (k_0+1) T_0 \right)}.$$

- Thus, for all $0 < \nu \leq \nu_0$; and all $t \in [0, T_0]$, we have $\|u(t) - \bar{u}(t)\|^2_{H^m} \leq (\nu t)\, \gamma_m$ ∎

- Note that we've proven convergence at a rate of $\sqrt{\nu}$. This is because we've measured the distance btw. u & $\bar{u}$ in H^m, while we knew that $\bar{u}$ is bounded in H^{m+1}.
- In fact [See Masmoudi 07] one may show that: if $\bar{u}_0 \in H^s$; $s > d/2+1$; and $s-2 < s' < s$ we have

$$\boxed{\sup_{t\in[0,T_0)} \| u(t) - \bar{u}(t) \|_{H^{s'}} \leq C \left((\nu t)^{\frac{(s-s')}{2}} + \| u_0 - \bar{u} \|_{H^{s'}} \right)}$$

C depens on $\|\bar{u}\|_{L^\infty(0,T_0;H^s)}$ & T_0.

- The convergence also holds in H^s, but at <u>no rate</u>. (this is harder to prove, see Kato '74 or Masmoudi '06)

- <u>Remark</u>: In the above case we see that the convergence rate may be of order $(\nu t)^{1-\varepsilon}$ $\forall\, \varepsilon > 0$, by simply allowing s to be sufficiently large.
- <u>Question</u>: do we have the same convergence rate when the initial datum is not in H^s, $s > d/2+1$.
  It turns out that in general the answer is No!
- As we have seen at the end of the last lecture, in two dimensions, if $\omega_0(x) = \bar{\omega}_0(|x|)$ is radially symmetric, then this is a stationary solution of Euler. On the other hand,

for NSE the solution's vorticity is given by the <u>heat</u> equation, $\omega(t) = e^{\nu t \Delta} \omega_0$; and is thus also radial.

- In particular, if $\bar{\omega}_0(r) = \chi_{[0,1]}$, i.e. we are dealing with a vortex patch supported on $B_1(0)$, then

$$\Big(\underset{\substack{\uparrow \\ \text{NSE}}}{\omega(t)} - \underset{\substack{\uparrow \\ \text{Euler}}}{\omega_0}\Big)(x) = \Big(\big(e^{\nu t \Delta} - 1\big)\, \omega_0\Big)(x)$$

$J_0(t) = \frac{1}{\pi}\int_0^\pi e^{it\cos\theta}\, d\theta$ is a Bessel function

and thus:

$$(\omega(t) - \omega_0)^\wedge(\xi) = \big(e^{-\nu t|\xi|^2} - 1\big)\, \hat{\omega}_0(\xi)$$

$$= \big(e^{-\nu t|\xi|^2} - 1\big)\, 2\pi \underbrace{\int_0^1 J_0(|\xi| r)\, r\, dr}_{\frac{J_1(|\xi|)}{|\xi|}}$$

which yields:

$$\|u(t) - u_0\|_{L^2}^2 = \|\hat{u}(t) - \hat{u}_0\|_{L^2}^2$$

$$= 2\pi \left\| \frac{i\xi^\perp}{|\xi|^3} \big(e^{-\nu t|\xi|^2} - 1\big)\, J_1(|\xi|) \right\|_{L^2}^2$$

$$\simeq (\nu t)^{3/2} \quad \text{(upper and lower bounds)}$$

so that $\quad \|u(t) - u_0\|_{L^2} \simeq (\nu t)^{3/4}$

- The rate $O((\nu t)^{1/2})$ was obtained first for vortex patches by Constantin-Wu ('95) and the sharp rate of $O((\nu t)^{3/4})$ was obtained by Abidi & Danchin ('04).

Remark: An example of "bad" inviscid limit, without boundaries.

- Consider the forced Euler and Navier-Stokes equations on $\mathbb{T}^2$, and we look only for steady states. More precisely, consider:

$$-\nu \Delta \omega + u \cdot \nabla \omega = g$$

where $-\Delta g = \lambda g$ for some $\lambda > 0$. This is called "Kolmogorov forcing".

- For all $\nu > 0$, we have the solution: $\boxed{\omega = \frac{1}{\nu \lambda} g}$ which follows since $u \cdot \nabla \omega = \nabla^\perp \psi \cdot \nabla(\Delta \psi) = \frac{1}{\nu \lambda^2} \nabla^\perp \psi \cdot \nabla \psi = 0.$

- As $\nu \to 0$, we see however that ω doesn't converge, in any topology; while it is not hard to show that the limiting problem $u \cdot \nabla \omega = g$ may have nontrivial solutions.

DAY 3

Kato criterion (1984)

Note Title 7/30/2015

- domain is half-plane $\mathbb{H} = \{(x_1, x_2) : x_2 > 0\}$
- (u^E, p^E) solve Euler: $\begin{cases} \partial_t u^E + (u^E \cdot \nabla) u^E + \nabla p^E = 0, & \text{in } \mathbb{H} \\ \nabla \cdot u^E = 0, & \text{in } \mathbb{H} \\ u^E \cdot n = -u_2^E = 0, & \text{on } \partial\mathbb{H} \end{cases}$
- denote $u_1^E\big|_{\partial\mathbb{H}} = \mathcal{U}^E(x_1, t)$, which is not necessarily zero.

- given smooth datum $u_0^E \in H_\sigma^s$; $s > d/2 + 1$
  we have local existence of a smooth solution on $[0, T)$ for some $T > 0$.
- we will denote by $C_{E,T}$ any constant that depends on $T > 0$, and on $\|u^E\|_{L^\infty(0,T;H^s)}$
- (u^{NS}, p^{NS}) solve Navier-Stokes: $\begin{cases} \partial_t u^{NS} - \nu \Delta u^{NS} + (u^{NS} \cdot \nabla) u^{NS} + \nabla p^{NS} = 0, & \text{in } \mathbb{H} \\ \nabla \cdot u^{NS} = 0, & \text{in } \mathbb{H} \\ u^{NS} = 0, & \text{on } \partial\mathbb{H} \end{cases}$
- assume $\|u_0^{NS} - u_0^E\|_{L^2} \le \varepsilon$

- define u^K to be a boundary layer corrector, which as of now will obey the following properties:
  (i) $u_1^K\big|_{\partial\mathbb{H}} = -\mathcal{U}^E$
  (ii) $u_2^K\big|_{\partial\mathbb{H}} = 0$

(iii) $\nabla \cdot u^k = 0$ in $\mathbb{H}$

(iv) u^k is supported in a strip of width $O(\alpha)$ near $\partial\mathbb{H}$, where $\alpha > 0$ is to be determined later on, in terms of ν.

(v) u^k is smooth, and for simplicity we may take it to be in H^s, just like Euler

- properties (iv) and (v) imply that

$$\sup_{t\in[0,T)} \| u^k(t) \|_{L^2(\mathbb{H})} = O(\alpha^{1/2})$$

and thus if $\alpha = \alpha(\nu) \to 0$ as $\nu \to 0$, we will also obtain that $\| u^k \|_{L^\infty(0,T; L^2(\mathbb{H}))} \to 0$ as $\nu \to 0$.

- let $v = u^{NS} - u^E - u^k$ be the error in approximating the Navier-Stokes flow with the Euler flow + the corrector flow u^k.

- note, by definition of u^k we have $\begin{cases} v = 0 & \text{on } \partial\mathbb{H} \\ \nabla \cdot v = 0 & \text{in } \mathbb{H} \end{cases}$

- We will do an energy estimate for v.

$$\partial_t (u^{NS} - u^E) - \nu \Delta u^{NS} + (u^{NS} - u^E) \cdot \nabla u^E + u^{NS} \cdot \nabla (u^{NS} - u^E) + \nabla (p^{NS} - p^E) = 0$$

$$\Downarrow$$

$$\partial_t v - \nu \Delta u^{NS} + v \cdot \nabla u^E + u^{NS} \cdot \nabla v \quad + \nabla(p^{NS} - p^E)$$

$$= \partial_t u^k - u^k \cdot \nabla u^E - u^{NS} \cdot \nabla u^k$$

$\Downarrow$

Here we use that $v|_{\partial H} = 0$; $\nabla \cdot v = 0$ and that

$$-\nu \int \Delta u^{NS} \cdot v = -\nu \int \Delta u^{NS} \cdot u^{NS} + \nu \int \Delta u^{NS} \cdot (u^E + u^k)$$

$$= \nu \|\nabla u^{NS}\|_{L^2}^2 - \nu \int \nabla u^{NS} \cdot \nabla (u^E + u^k)$$

where in the last equality we used that $u^E + u^k = 0$ on ∂H

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \nu \|\nabla u^{NS}\|_{L^2}^2$$

$$= -\int (v \cdot \nabla u^E) \cdot v$$

$$+ \nu \int \nabla u^{NS} \cdot \nabla u^E$$

$$+ \nu \int \nabla u^{NS} \cdot \nabla u^k$$

$$+ \int \partial_t u^k \cdot v - \int (u^k \cdot \nabla u^E) \cdot v$$

$$- \int (u^{NS} \cdot \nabla u^k) \cdot u^{NS} - \int (u^{NS} \cdot \nabla u^E) \cdot u^k$$

$$\leq \|\nabla u^E\|_{L^\infty} \|v\|_{L^2}^2 + \frac{\nu}{4} \|\nabla u^{NS}\|_{L^2}^2 + \nu \|\nabla u^E\|_{L^2}^2$$

$$+ \|\partial_t u^k\|_{L^2} \|v\|_{L^2} + \|u^k\|_{L^2} \|\nabla u^E\|_{L^\infty} \|v\|_{L^2} + \|u^{NS}\|_{L^2} \|\nabla u^E\|_{L^\infty} \|u^k\|_{L^2}$$

$$+ \nu \int \partial_j u_i^{NS} \cdot \partial_j u_i^k - \int u_j^{NS} \cdot \partial_j u_i^k \cdot u_i^{NS}$$

- at this point we observe that
  - $\to \|\partial_1 u^k\|_{L^2} \sim \|u^k\|_{L^2} \sim O(\alpha^{1/2})$ since u^k should vary on $O(1)$ scales in x_1
  - $\to$ thus, only $\partial_2 u_1^k$ should be a bad term, since $\partial_2 u_2^k = -\partial_1 u_1^k$ by the div-free condition
- thus:

$$\nu \int \partial_j u_i^{NS} \cdot \partial_j u_i^k \leq \nu \int \partial_2 u_1^{NS} \cdot \partial_2 u_1^k + \frac{\nu}{4} \|\nabla u^{NS}\|_{L^2}^2 + \nu \|\partial_1 u^k\|_{L^2}^2$$

- also:

$$-\int u_j^{NS} \cdot \partial_j u_i^k \cdot u_i^{NS} \leq -\int u_2^{NS} \, \partial_2 u_1^k \, u_1^{NS} + C \|\partial_1 u^k\|_{L^2(\Gamma_\alpha)} \|u^{NS}\|^2_{L^4(\Gamma_\alpha)}$$

$$\leq -\int u_2^{NS} \, \partial_2 u_1^k \, u_1^{NS} + C \|\partial_1 u^k\|_{L^2} \|u^{NS}\|_{L^2} \|\nabla u^{NS}\|_{L^2(\Gamma_\alpha)}$$

2D Sobolev embedding $L^4 \sim (L^2)^{1/2} (H^1)^{1/2}$ (CHECK!)

- Recall that $\|u^{NS}(t)\|_{L^2} \leq \|u_0^{NS}\|_{L^2} \leq 1 + \|u_0^E\|_{L^2}$.
- In summary, denoting by C_E constants that depend on u^E on $[0,T]$:

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2} + \frac{\nu}{4}\|\nabla u^{NS}\|^2_{L^2}$$

$$\leq C_E \|v\|^2_{L^2} + \|v\|_{L^2} \left(\|\partial_t u^k\|_{L^2} + C_E \|u^k\|_{L^2} \right)$$

$$+ C_E \nu + C_E \|u^k\|_{L^2} + C_E \|\partial_1 u^k\|_{L^2} \|\nabla u^{NS}\|_{L^2(\Gamma_\alpha)}$$

$$+ \nu \int \partial_2 u_1^{NS} \, \partial_2 u_1^k - \int u_2^{NS} \, \partial_2 u_1^k \, u_1^{NS}$$

- At this stage we bound the last two terms in the energy estimate.
- First, since we expect u^k to be supported in a layer $\Gamma_\alpha = \{(x_1, x_2) : 0 < x_2 \leq \alpha\}$ near $\mathbb{H}$, we see that $\partial_2 u^k$ is also supported in Γ_α.

  Thus,

$$\left| \nu \int_{\mathbb{H}} \partial_2 u_1^{NS} \, \partial_2 u_1^k \right| \leq \nu \|\partial_2 u_1^{NS}\|_{L^2(\Gamma_\alpha)} \|\partial_2 u_1^k\|_{L^2(\Gamma_\alpha)}$$

- Secondly,

$$\left| \int u_2^{NS} \, \partial_2 u_1^k \, u_1^{NS} \right| \leq \| u^{NS} \|_{L^2(\Gamma_\alpha)}^2 \, \| \partial_2 u_1^k \|_{L^\infty}$$

$$\leq C \alpha^2 \, \| \partial_2 u^{NS} \|_{L^2(\Gamma_\alpha)}^2 \, \| \partial_2 u_1^k \|_{L^\infty}$$

here we used FTC: since $u^{NS}(x_1, 0) = 0$!

$$\| u^{NS} \|_{L^2(\Gamma_\alpha)}^2 = \int_{\Gamma_\alpha} |u^{NS}(x_1, x_2)|^2 \, dx_1 dx_2 = \int_{\Gamma_\alpha} \left| \int_0^{x_2} \partial_2 u^{NS}(x_1, z) \, dz \right|^2 dx_1 dx_2$$

$$\leq \int_0^\alpha \int_{\mathbb{R}} \left(x_2 \int_0^{x_2} | \partial_2 u^{NS}(x_1, z) |^2 \, dz \right) dx_1 dx_2$$

$$\leq \alpha^2 \, \| \partial_2 u^{NS} \|_{L^2(\Gamma_\alpha)}^2$$

- In summary:

$$\frac{1}{2} \frac{d}{dt} \| v \|_{L^2}^2 + \frac{\nu}{4} \| \nabla u^{NS} \|_{L^2}^2$$

$$\leq C_E \| v \|_{L^2}^2 + \| v \|_{L^2} \left(\| \partial_t u^k \|_{L^2} + C_E \| u^k \|_{L^2} \right)$$

$$+ C_E \nu + C_E \| u^k \|_{L^2} + C_E \| \partial_1 u^k \|_{L^2} \, \| \nabla u^{NS} \|_{L^2(\Gamma_\alpha)}$$

$$+ \nu \| \nabla u^{NS} \|_{L^2(\Gamma_\alpha)} \, \| \partial_2 u_1^k \|_{L^2} + C \alpha^2 \, \| \nabla u^{NS} \|_{L^2(\Gamma_\alpha)}^2 \, \| \partial_2 u_1^K \|_{L^\infty}$$

- At this stage we claim that we may choose u^k s.t.

$$\begin{cases} \| u^k \|_{L^2} + \| \partial_1 u^k \|_{L^2} + \| \partial_t u^k \|_{L^2} \leq C_E \, \alpha^{1/2} \\ \| \partial_2 u_1^k \|_{L^2} \leq C_E \, \alpha^{-1/2} \\ \| \partial_2 u_1^k \|_{L^\infty} \leq C_E \, \alpha^{-1} \end{cases}$$

- Assuming for the moment this actually can be done, we obtain the estimate

$$\frac{d}{dt}\|v\|_{L^2}^2 + \frac{\nu}{2}\|\nabla u^{NS}\|_{L^2}^2$$

$$\leq C_E\|v\|_{L^2}^2 + C_E(\nu+\alpha+\alpha^{1/2})$$

$$+ C_E\left(\nu\alpha^{-1/2}+\alpha^{1/2}\right)\|\nabla u^{NS}\|_{L^2(\Gamma_\alpha)}$$

$$+ \underbrace{C_E\,\alpha\,\|\nabla u^{NS}\|_{L^2(\Gamma_\alpha)}^2}$$

↳ this last term, at best may be absorbed on the LHS

↳ In turn, this enforces a constraint on α:

$$C_E\,\alpha = \frac{\nu}{2}$$

- We then obtain:

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq C_E\|v\|_{L^2}^2 + \tilde{C}_E\left(\nu+\nu^{1/2}\right) + \tilde{C}_E\,\nu^{1/2}\|\nabla u^{NS}\|_{L^2(\Gamma_{c\nu})}$$

- Recalling that

$$\|v_0\|_{L^2} \leq \|u_0^{NS}-u_0^E\|_{L^2} + \|u_0^k\|_{L^2}$$

$$\leq \|u_0^{NS}-u_0^E\|_{L^2} + \tilde{C}_E\,\nu^{1/2}$$

we obtain Kato's conditions:

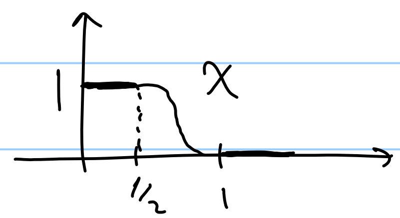
If
- $\|u_0^{NS}-u_0^E\|_{L^2} \to 0$ as $\nu\to 0$
- $\int_0^T \nu^{1/2}\|\nabla u^{NS}\|_{L^2(\Gamma_{c\nu})}\,dt \to 0$ as $\nu\to 0$

THEN $\|u^{NS}-u^E\|_{L^\infty(0,T;L^2(\mathbb{H}))} \to 0$ as $\nu\to 0$

- An example of a corrector u^K as desired may be given as

$$u^K = \nabla^\perp \left(x_2 \, \overline{u}^E(x_1,t) \, \chi\left(\frac{x_2}{\alpha}\right) \right)$$

where



- <u>Note</u>: Kato's condition is if and only if. Indeed,

for $\nu > 0$: $$\nu \int_0^T \|\nabla u^{NS}\|_{L^2}^2 \, dt \leq \|u_0^{NS}\|_{L^2}^2 - \|u^{NS}(T)\|_{L^2}^2$$

if $\|u_0^{NS} - u_0^E\|_{L^2} \to 0$ as $\nu \to 0$

and $\|u^{NS}(t) - u^E(t)\|_{L^2} \to 0$ as $\nu \to 0$ for a.e. $t \in [0,T]$,

then
$$\limsup_{\nu \to 0} \left(\|u_0^{NS}\|_{L^2}^2 - \|u^{NS}(T)\|_{L^2}^2 \right)$$

$$\leq \lim_{\nu \to 0} \|u_0^{NS}\|_{L^2}^2 - \liminf_{\nu \to 0} \|u^{NS}(T)\|_{L^2}^2$$

$$= \|u_0^E\|_{L^2}^2 - \|u^E(T)\|_{L^2}^2$$

$= 0$, for smooth solutions of Euler.

<u>Remark 1</u>: (Masmoudi '98) if $\nu \Delta u$ is replaced by $\nu \partial_{xx} u + \eta \partial_{yy} u$,

then the above proof may be modified to yield that

$\Big\{$ in the limit $\nu \to 0$; $\eta \to 0$; $\frac{\eta}{\nu} \to 0$;

the anisotropic viscosity Navier-Stokes solution converges to the Euler solution in $L^2(\mathbb{H})$.

- To see this, note that the dangerous terms are

$$\underbrace{\eta \int \partial_2 u_1^{NS}\, \partial_2 u_1^{k}\, dx}_{} \qquad \text{and} \qquad \underbrace{\int u_2^{NS}\, \partial_2 u_1^{k}\, u_1^{NS}}_{}$$

$$\begin{aligned}
&\leq \eta \|\partial_2 u^{NS}\|_{L^2} \|\partial_2 u_1^k\|_{L^2} \\
&\leq \frac{\eta}{4} \|\partial_2 u^{NS}\|_{L^2}^2 + C\eta \|\partial_2 u_1^k\|_{L^2}^2 \\
&\leq \frac{\eta}{4} \|\partial_2 u^{NS}\|_{L^2}^2 + C_E \frac{\eta}{\alpha}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha^2 \|\partial_2 u_2^{NS}\|_{L^2(\Gamma_\alpha)} \|\partial_2 u_1^{NS}\|_{L^2(\Gamma_\alpha)} \|\partial_2 u_1^k\|_{L^\infty(\Gamma_\alpha)} \\
&\leq C_E\, \alpha \|\partial_1 u_1^{NS}\|_{L^2} \|\partial_2 u_1^{NS}\|_{L^2} \\
&\leq \frac{\nu}{4} \|\partial_1 u^{NS}\|_{L^2}^2 + \left(C_E \frac{\alpha^2}{\nu\eta}\right) \cdot \eta \|\partial_2 u_1^{NS}\|_{L^2}^2
\end{aligned}$$

- Thus if we ensure $\frac{\eta}{\alpha} \to 0$ and $\frac{\alpha^2}{\nu\eta} \leq \frac{1}{C_E}$ we are done.

- Letting $\alpha = \eta^{\theta} \nu^{1-\theta} \iff \underbrace{\left(\frac{\eta}{\nu}\right)^{1-\theta} \to 0 \quad \& \quad \left(\frac{\eta}{\nu}\right)^{2\theta-1} \to 0}_{}$

which is ok when $\eta/\nu \to 0$

by letting $\theta \in (1/2, 1)$!

<u>Remark 2:</u> Temam-Wong ('97) (only tangential gradient)

Let $\beta \in [3/4, 1)$. If we know that

$$\nu^{\beta - 1/2} \|\partial_1 u_1^{NS}\|_{L^2(0,T;L^2(\Gamma_{\nu^\beta}))} \to 0 \quad \text{as } \nu \to 0$$

Then $\|u^{NS} - u^E\|_{L^\infty(0,T;L^2)} \to 0$ as $\nu \to 0$.

Proof: As before, we consider only the dangerous terms:

$$\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla u^{NS}\|_{L^2}^2 \leq C\nu\|\nabla u^k\|_{L^2}^2 + o(1) + C\|v\|_{L^2}^2$$
$$+ \left|\int u^{NS}\cdot\nabla u^k\, u^{NS}\right|$$
$$\leq o(1) + C\|v\|_{L^2}^2 + \underbrace{C\nu^{1-\beta}}_{\text{ok if } \beta<1} + \left|\int u^{NS}\cdot\nabla u^k\, u^{NS}\right|$$

Now: • $\left|\int u_1^{NS}\,\partial_1 u^k\cdot u^{NS}\right| \leq \|\partial_1 u^k\|_{L^\infty}\|u^{NS}\|_{L^2(\Gamma_{\nu\beta})}^2$

$$\leq \underbrace{C\nu^{2\beta}}_{\text{ok if } 2\beta>1}\|\nabla u^{NS}\|_{L^2}^2$$

• $\left|\int u_2^{NS}\,\partial_2 u_2^k\, u_2^{NS}\right| = \left|\int u_2^{NS}\,\partial_1 u_1^k\, u_2^{NS}\right| \Rightarrow$ ok as before

• $\left|\int u_2^{NS}\,\partial_2 u_1^k\, u_1^{NS}\right|$

$$= \left|\int \partial_2 u_2^{NS}\, u_1^k\, u_1^{NS}\right| + \left|\int u_2^{NS}\, u_1^k\,\partial_2 u_1^{NS}\right|$$

$$\leq C\|\partial_1 u_1^{NS}\|_{L^2(\Gamma_{\nu\beta})}\|u_1^{NS}\|_{L^2(\Gamma_{\nu\beta})} + \|u_2^{NS}\|_{L^2(\Gamma_{\nu\beta})}\|\partial_2 u_1^{NS}\|_{L^2(\Gamma_{\nu\beta})}$$

$$\leq C\nu^{\beta}\|\nabla u^{NS}\|_{L^2}\|\partial_1 u_1^{NS}\|_{L^2(\Gamma_{\nu\beta})}$$

$$\leq \frac{\nu}{2}\|\nabla u^{NS}\|_{L^2}^2 + \underbrace{\nu^{2\beta-1}\|\partial_1 u_1^{NS}\|_{L^2(\Gamma_{\nu\beta})}^2}_{\text{by assumption } \to 0 \text{ as } \nu\to 0.}$$

∎

Remark 3: (Constantin - Kukavica - V. '14) (Signed Kato condition).

Assume $\nabla^E(x_1,t) \geq 0$ on $[0,T]$ (condition ensured by datum) and that

$$\nu^{1/2} \left\| \left(\omega^{NS} + \frac{M_\nu(t)}{\nu} \right)_- \right\|_{L^2(\tilde{\Gamma}_\nu)} \leq M_\nu(t)$$

only the very negative vorticity

where $\int_0^T M_\nu(t)\, dt \to 0$ as $\nu \to 0$

and $\tilde{\Gamma}_\nu = \left\{ (x_1, x_2) : 0 < x_2 < \frac{\nu t}{C} \log \frac{C}{M_\nu(t)} \right\}$

with $C = C(\|u^E\|_{L^\infty_t H^s_x})$.

Then $\|u^{NS} - u^E\|_{L^2(\mathbb{H})} \to 0$, uniformly on $[0,T]$.

Q: does this mean vorticity may blow up at the boundary?

Remark 4: (Kelliher '08) (vortex sheet on boundary)

We have $\|u^{NS} - u^E\|_{L^2(\mathbb{H})} \to 0$ uniformly on $[0,T]$

iff $u^{NS} \to u^E$ weakly in $L^2_\sigma(\mathbb{H})$, uniformly on $[0,T]$ $\boxed{\text{Kato}}$

iff $\omega^{NS} \to \omega^E - u_1^E|_{\partial\mathbb{H}}\, \mu$ in $(H^1(\mathbb{H}))'$, uniformly on $[0,T]$

- $\mu \in \mathcal{M}(\bar{\Omega})$ ← finite Borel signed measure on $\bar{\Omega}$
- μ is supported on $\partial\mathbb{H}$
- $\mu|_{\partial\mathbb{H}}$ = arc length, i.e. lebesgue measure

Note: the last point implies $\omega^{NS} \to \omega^E$ in $H^{-1}(\mathbb{H})$, unif. on $[0,T]$

H^{-1}: dual of $H^1_0(\mathbb{H})$

which in turn implies $u^{NS} \to u^E$ weakly in $L^2(\mathbb{H})$,

which in turn implies through energy ineq. that inviscid limit holds

Thus: we only need to prove that if the inviscid limit holds in L^2, then $\omega^{NS} \to \omega^E - \mathcal{U}^E \mu$ in $(H^1(\mathbb{H}))'$.

To see the latter, we let $\varphi \in H^1(\mathbb{H})$. (Not $H^1_0(\mathbb{H})$!)

Then,
$$\langle \omega^{NS}(t), \varphi \rangle = \langle \nabla^\perp \cdot u^{NS}(t), \varphi \rangle$$
$$= -\langle u^{NS}(t), \nabla^\perp \varphi \rangle \qquad \left(\text{No B.C. since } u^{NS}\big|_{\partial \mathbb{H}} = 0\right)$$
$$\Big\downarrow \text{(by assumption)}$$
$$= -\langle u^E(t), \nabla^\perp \varphi \rangle \qquad \text{uniformly on } [0,T].$$

But,
$$-\langle u^E, \nabla^\perp \varphi \rangle = \langle \nabla^\perp \cdot u^E, \varphi \rangle - \int_{\partial \mathbb{H}} \underbrace{u^E \cdot n^\perp}_{= \mathcal{U}^E(x,t)} \varphi\big|_{\partial \mathbb{H}} \, dx$$
$$= \langle \omega^E, \varphi \rangle - \langle \mathcal{U}^E \mu, \varphi \rangle \qquad \blacksquare$$

<u>Remark 5</u>: In fact (see also Remark 3), the $L^\infty([0,T], L^2(\mathbb{H}))$ inviscid limit turns out to be equivalent to
$$\lim_{\nu \to 0} \nu \int_0^T \int_{\partial \mathbb{H}} \omega^{NS} \, \mathcal{U}^E \, dx \, dt = 0$$

<u>Notes</u>:
- Constantin-Kukavica-Vicol ('14) show that you only need ≤ 0 for the inviscid limit to hold.
- Kelliher ('14) shows that in fact
$$\nu \int_0^T \int_{\partial \mathbb{H}} \omega^{NS} \varphi \, dx \, dt \to 0 \qquad \forall \varphi \in C^1([0,T] \times \partial \mathbb{H})$$
is necessary and sufficient

- Bardos – Titi ('13) show that

$$\nu\, \omega^{NS} \to 0 \qquad \text{in} \quad \mathcal{D}'([0,T] \times \partial\mathbb{H})$$

is necessary & sufficient.

<u>Prove:</u> $\nu \int_0^T \int_{\partial\mathbb{H}} \omega^{NS} \, \overline{u}^E \to 0$ implies inviscid limit

<u>Necessity</u> test the equations against Kato-style lift of a boundary $C^1_{t,x}$ function.

<u>Remark 6:</u> 2D NSE with vorticity odd in y $\Rightarrow$ both obey the cond. in Remark 5

<u>Remark 7:</u> Shear flow

DAY 4

Prandtl derivation and basics

Note Title 8/3/2015

- For the sake of simplicity, let $\Omega = \mathbb{R}^2_+$ (simple boundary) (the issue of regularity is not there for an $O(1)$ interval, and thus, we may as well work in 2D)

- Write $\mathbb{H} = \{ (x,y) : y > 0 \}$ $\Rightarrow$ $n = (0,-1)$ is outward unit normal

  Navier - Stokes solution : (u^{NS}, v^{NS}) ; p^{NS}

  Euler solution : (u^E, v^E) ; p^E

- Recall that :

  → Navier-Stokes obeys Dirichlet boundary conditions

  $$u^{NS}(x,0,t) = v^{NS}(x,0,t) = 0$$

  → Euler obeys slip boundary conditions

  $$v^E(x,0,t) = 0 \qquad \left((u^E, v^E) \cdot n = -v^E \right)$$

  We will denote $u^E(x,0,t) = U^E(x,t)$

- Since $U^E \neq 0$ (if so initially), we have a mismatch in boundary conditions, and we may look as before, for a boundary layer corrector.

  this is what Prandtl did in 1904

- Recall that for $u_0^{NS}, u_0^E \in H^s$; $s > d/2 + 1$ both solutions exist (globally in 2D) on an $O(1)$ w.r.t. time interval.

- Also, recall that for the inviscid limit to even make sense, we need

$$\|u_0^{NS} - u_0^E\|_{L^2} = o(1) \text{ w.r.t. } \nu, \quad \text{as } \nu \to 0.$$

- The boundary layer corrector is thus only "needed" for u^{NS} (the x- component of the velocity)

- Prandtl made the ANSATZ: (let $Y = y/\delta$)

$$\begin{cases} u^{NS}(x,y,t) = u^E(x,y,t) + u^P(x, y/\delta, t) \\ u^P(x,0,t) = -\overline{U}^E(x,t) \\ u^P(x,Y,t) \to 0 \quad \text{as } Y \to \infty \end{cases}$$

- As before (in the proof of Kato), we may define the Prandtl corrector as an incompressible vector field. Then we need to define the second component as the Prandtl field as:

$$\delta\, v^P(x, y/\delta, t)$$

where

$$\begin{cases} \partial_x u^P + \partial_Y v^P = 0; \\ v^P\big|_{Y=0} = 0 \end{cases}$$

<u>Note</u>: in fact v^P is <u>NOT</u> an unknown, we have:

$$\boxed{v^P(x,Y,t) = -\int_0^Y \partial_x u^P(x,z,t)\, dz}$$

- Thus, we make Prandtl's ansatz: THIS is HARD TO JUSTIFY

$$\boxed{\begin{aligned} \begin{pmatrix} u^{NS} \\ v^{NS} \end{pmatrix}(x,y,t) &= \begin{pmatrix} u^{E} \\ v^{E} \end{pmatrix}(x,y,t) + \begin{pmatrix} u^{P} \\ \delta v^{P} \end{pmatrix}(x, y/\delta, t) + O(\delta) \\ p^{NS}(x,y,t) &= p^{E}(x,y,t) + p^{P}(x, y/\delta, t) + O(\delta) \end{aligned}}$$

- Let's first look at the v^{NS} equation.

$$\begin{aligned} \partial_t (v^E + \delta v^P) &+ (u^E \partial_x + v^E \partial_y)(v^E + \delta v^P) \\ &+ (u^P \partial_x + \delta v^P \partial_y)(v^E + \delta v^P) \\ &+ \frac{1}{\delta}(\partial_Y p^P) + \partial_y p^E = \nu \Delta (v^E + \delta v^P) \end{aligned}$$

where we have denoted $Y = y/\delta$.

Using the equation obeyed by v^E, we are left with:

$$\left\{ \begin{aligned} &\delta \left(\partial_t v^P + u^E \partial_x v^P + u^P \partial_x v^P + v^P \partial_y v^E + v^P \partial_Y v^P - \nu \partial_x^2 v^P \right) \\ &- \nu \Delta v^E \\ &+ \left(v^E \partial_Y v^P + u^P \partial_x v^E \right) + (\nu \delta^{-1}) \partial_Y^2 v^P \\ &+ \boxed{\frac{1}{\delta}(\partial_Y p^P)} = 0 \end{aligned} \right.$$

as we shall see next, this is the dominant term

- Let's now look at the u^{NS} equation:

$$\begin{aligned} \partial_t (u^E + u^P) &+ (u^E \partial_x + v^E \partial_y)(u^E + u^P) \\ &+ (u^P \partial_x + \delta v^P \partial_y)(u^E + u^P) \\ &+ \partial_x (p^E + p^P) = \nu \Delta (u^E + u^P) \end{aligned}$$

Using the equation obeyed by u^E, we are left with:

$$\begin{cases} \partial_t u^P + u^E \partial_x u^P + \frac{1}{\delta} v^E \partial_Y u^P \\ \quad + u^P \partial_x u^E + u^P \partial_x u^P \\ \quad + v^P \partial_Y u^P + \partial_x p^P - \frac{\nu}{\delta^2} \partial_{YY} u^P \\ + \delta (v^P \partial_y u^E) + \nu \Delta u^E - \nu \partial_{xx} u^P = 0 \end{cases}$$

- First, note that $v^E \big|_{y=0} = 0$ and that $\partial_Y u^P$ should be roughly speaking supported on $|y| \lesssim \delta$.

  Thus, to leading order, for $|y| \lesssim \delta$ we have:

$$\begin{aligned} v^E(x,y,t) &= \int_0^y \partial_y v^E(x,z,t)\, dz \\ &= - \int_0^y \partial_x u^E(x,z,t)\, dz \\ &= -y\, \partial_x U^E(x,t) - \int_0^y \int_0^z \left(\partial_x \partial_w u^E(x,w,t) \right) dw\, dz \end{aligned}$$

$$\Rightarrow \frac{1}{\delta} v^E \partial_Y u^P = -Y\, \partial_x U^E \partial_Y u^P - \frac{1}{\delta} \underbrace{\left(\int_0^y \int_0^z \partial_x \partial_w u^E(x,w,t)\, dw\, dz \right)}_{\simeq\, y^2/\delta \,=\, yY \,\simeq\, \delta Y} \partial_Y u^P$$

due to support near $y \simeq \delta$

- Next, we see that the dissipative term ∂_{YY} is $O(1)$ iff we choose $\boxed{\delta = \sqrt{\nu}}$

- This is the main point in <u>Prandtl's theory</u>; that all the action is happening within an $O(\sqrt{\nu})$ layer near the boundary of the domain.
- With this notation, let's read off the leading order equations:

$$
\begin{cases}
\partial_t u^P + U^E \partial_x u^P - Y \partial_x U^E \, \partial_Y u^P \\
\quad + u^P \partial_x U^E + u^P \partial_x u^P \\
\quad + v^P \partial_Y u^P + \partial_x p^P - \partial_{YY} u^P = 0
\end{cases}
$$

$$
\begin{cases}
\partial_Y p^P = 0
\end{cases}
$$

with errors:

$$
\begin{aligned}
F_u = & \; Y \, \partial_x u^P(x,Y,t) \cdot \frac{1}{y}\int_0^y \partial_z u^E(x,z,t)\, dz \cdot \sqrt{\nu} \\
& + Y \, u^P(x,Y,t) \cdot \frac{1}{y}\int_0^y \partial_x \partial_z u^E(x,z,t)\, dz \cdot \sqrt{\nu} \\
& - Y^2 \partial_Y u^P(x,Y,t) \cdot \frac{1}{y^2}\int_0^y \int_0^z \partial_x \partial_w u^E(x,w,t)\, dw\, dz \cdot \sqrt{\nu} \\
& + v^P(x,Y,t) \; \partial_y u^E(x,y,t) \cdot \sqrt{\nu} \\
& - \partial_{xx} u^P(x,Y,t) \cdot \nu \\
& + \Delta u^E(x,y,t) \cdot \nu
\end{aligned}
$$

(which one may "hope" is $O(\sqrt{\nu})$ as $\nu \to 0$)

and

$$F_v = \sqrt{\nu}\left(\partial_t v^P + u^E\partial_x v^P + u^P\partial_x v^P + v^P\partial_y v^E + v^P\partial_Y v^P + \partial_Y^2 v^P\right)$$

$$-\nu^{3/2}\partial_x^2 v^P - \nu\Delta v^E$$

$$-\sqrt{\nu}\, Y\partial_y v^P \quad \frac{1}{y}\int_0^y \partial_x u^E(x,z,t)\,dz$$

$$-\sqrt{\nu}\, Y u^P(x,Y,t)\frac{1}{y}\int_0^y \partial_x u^E(x,z,t)\,dz$$

(which we may again hope to be $O(\sqrt{\nu})$)

- Recall that in the above expression v^P is in fact determined from u^P in a nonlocal way, w.r.t. Y.

- Thus, up to a postulated $O(\sqrt{\nu})$ error, we have arrived at the Prandtl equations:

absent

$$\partial_t u^P - \partial_Y^2 u^P + u^P\partial_x u^P + v^P\partial_Y u^P + \partial_x p^P$$
$$+ U^E\partial_x u^P - Y\,\partial_x U^E\,\partial_Y u^P + u^P\partial_x U^E = 0$$

$$\partial_Y p^P = 0 \implies p^P = p^P(x,t)$$ $\Big\{$ is taken to be a constant in x as well $\Big\}$

$$u^P\big|_{Y=0} = -U^E$$

$$u^P\big|_{Y=\infty} = 0$$

$$v^P = -\int_0^Y \partial_x u^P$$

Note: you may typically see the Prandtl equations written in the variables

$$\begin{cases} \bar{u}^P(x,Y,t) = U^E(x,t) + u^P(x,Y,t) \\ \bar{v}^P(x,Y,t) = -Y\,\partial_x U^E(x,t) + v^P(x,Y,t) = -\int_0^Y \partial_x \bar{u}^P \\ \partial_x \bar{p}^P(x,t) = -\partial_t U^E(x,t) - U^E(x,t)\,\partial_x U^E(x,t) \boxed{+\underbrace{\partial_x p^P}_{=0}} \end{cases}$$

Then: $\left\{ \begin{array}{l} \bar{u}^P\big|_{Y=0} = 0 \\ \\ \bar{u}^P\big|_{Y\to\infty} = U^E(x,t) \end{array} \right\}$ are the new boundary conditions, and

the new Prandtl equations are:

$$\partial_t \bar{u}^P - \partial_Y^2 \bar{u}^P + (\bar{u}^P - U^E)\,\partial_x(\bar{u}^P - U^E) + (\bar{v}^P + Y\partial_x U^E)\,\partial_Y \bar{u}^P + \partial_x \bar{p}$$
$$+ U^E(\partial_x \bar{u}^P - \partial_x U^E) - Y\,\partial_x U^E \partial_Y \bar{u}^P + (\bar{u}^P - U^E)\partial_x U^E$$
$$- \partial_t U^E + \partial_t U^E + U^E \partial_x U^E = 0$$

$$\Rightarrow \boxed{\partial_t \bar{u}^P - \partial_Y^2 \bar{u}^P + \bar{u}^P \partial_x \bar{u}^P + \bar{v}^P \partial_Y \bar{u}^P = -\partial_x \bar{p}}$$

- Question 1: does $\|(F_u, F_v)\|_X = O(\sqrt{\nu})$ or at least $o(1)$ as $\nu \to 0$ w.r.t. some norm X?

If YES, we say the Prandtl expansion is justified.

- <u>Note</u>: if Question 1 is YES, it automatically implies that $\|(u^{NS}, v^{NS}) - (u^{E}, v^{E})\|_{L^2(\#1)} \to 0$ as $\nu \to 0$.

- <u>Note</u>: if Question 1 is NO, it doesn't mean the inviscid limit doesn't hold, it just means that the Prandtl expansion is <u>not justified</u>.

- <u>Question 2</u>: in what spaces X are the Prandtl equations locally well-posed? Without this, we don't have a hope of even tackling Question !

- <u>Question 3</u>: if the equations are locally well-posed, do they blow-up in finite time, or do they live forever?

Blowup in Prandtl (E & Engquist '97)

- Consider $P^E, U^E = 0$.
- Then 2D Prandtl becomes:

$$\begin{cases} \partial_t u + u u_x + v u_y = u_{yy} \\ v = -\int_0^y u_x \\ u|_{y=0} = u|_{y=\infty} = 0 \end{cases}$$

- Assume the solution u stays in $L^\infty_{loc}([0,\infty); H^s)$.
  Thus, this will be a proof by contradiction.
- We assume $\begin{cases} u_0(x,y) \text{ is odd in } x. \\ v_0(x,y) \text{ is even in } x. \end{cases}$
- $\partial_t u \to$ odd in x
  $u u_x \to$ odd $\cdot$ even $=$ odd in x
  $v \partial_y u \to$ even $\cdot$ odd $=$ odd in x
  $\partial_y^2 u \to$ odd in x

  $\Rightarrow$ as long as the solution u stays smooth (C^1_x) and unique, it must be odd in x!
- We may thus write the solution as

$$\begin{cases} u(x,y,t) = -x\, b(x,y,t) \\ v(x,y,t) = \int_0^y \big(b(x,z,t) + x\,\partial_x b(x,z,t)\big)\, dz \end{cases}$$

for some, also smooth, $b(x,y,t)$.
- The equation obeyed by b is:

$$\partial_t b - \partial_y^2 b = b(b + x\,\partial_x b) - v\,\partial_y b$$

with boundary conditions $b|_{y=0} = b|_{y=\infty} = 0$.

- The next trick is to restrict the above equation to $x=0$.

  Let $a(y,t) = b(0,y,t)$.

  Then, we obtain:

$$\begin{cases} \partial_t a - \partial_y^2 a = a^2 - \left(\int_0^y a\, dz\right) \cdot \partial_y a \\ a|_{y=0} = 0 \quad ; \quad a|_{y=\infty} = 0. \end{cases}$$

- The initial datum $a_0(y) = b_0(0,y) = -(\partial_x u_0)(0,y)$, is still for us to choose.
- The theorem then is:

$$\begin{cases} \text{Let } a_0 \geq 0 \; ; \; a_0 \in C_0^\infty(\mathbb{R}_+) \; ; \text{ such that } E(a_0) < 0, \\ \text{where } \quad E(a) = \int_0^\infty \left[\frac{1}{2}(\partial_y a)^2 - \frac{1}{4}(a^3)\right] dy. \\ \text{Then the solution cannot remain bounded in } W^{2,4}(\mathbb{R}_+) \end{cases}$$

Proof: Let $F(t) = |a|_{L^2}^2$.

We have:
$$\frac{d}{dt} |a|_{L^2}^2 = -2|\partial_y a|_{L^2}^2 + 3|a|_{L^3}^3$$
$$= -4E + 2|a|_{L^3}^3$$

where recall that $E = \frac{1}{2}|\partial_y a|_{L^2}^2 - \frac{1}{4}|a|_{L^3}^3$

- We next compute $\frac{dE}{dt}$. We have

$$\frac{d}{dt}|\partial_y a|_{L^2}^2 = -2|\partial_y^2 a|_{L^2}^2 + 3\int (\partial_y a)^2 a$$
$$\frac{d}{dt}|a|_{L^3}^3 = 4|a|_{L^4}^4 - 6\int (\partial_y a)^2 a$$

- Now it is clear why we defined $E(a)$ as such:

$$\frac{d}{dt} E(a) = -|\partial_y^2 a|_{L^2}^2 - |a|_{L^4}^4 \leq 0.$$

- Thus, under the assumption that $E(a_0) < 0$, we may ensure that $E(a) < 0 \quad \forall t \geq 0$.
- Next, we prove that $\dfrac{(-E(a))}{F(a)}$ is increasing in time!

$$\frac{d}{dt}\left(-\frac{E}{F}\right) = \frac{1}{F^2}\left(E\frac{dF}{dt} - F\frac{dE}{dt}\right)$$

But:

$$E\frac{dF}{dt} - F\frac{dE}{dt} = \left(\frac{1}{2}|\partial_y a|_{L^2}^2 - \frac{1}{4}|a|_{L^3}^3\right)\left(-2|\partial_y a|_{L^2}^2 + 3|a|_{L^3}^3\right)$$
$$+ |a|_{L^2}^2\left(|\partial_y^2 a|_{L^2}^2 + |a|_{L^4}^4\right)$$

$$= \underbrace{\left(-|\partial_y a|_{L^2}^4 + |a|_{L^2}^2 |\partial_y^2 a|_{L^2}^2\right)}_{\geq 0}$$
$$+ 3|\partial_y a|_{L^2}^2 |a|_{L^3}^3$$
$$+ \underbrace{\left(-\frac{3}{4}|a|_{L^3}^6 + |a|_{L^2}^2 |a|_{L^4}^4\right)}_{\geq 0.} \qquad \geq 0.$$

- Thus, $-E(a) \geq F(a)\left(\dfrac{-E(a_0)}{F(a_0)}\right)$

and thus:

$$2|a|_{L^3}^3 + 4F(a)\left(\frac{-E(a_0)}{F(a_0)}\right) \leq \frac{dF}{dt} \leq 3|a|_{L^3}^3 - 2|\partial_y a|_{L^2}^2$$

$$\Rightarrow \quad |\partial_y a|_{L^2}^2 \leq \tfrac{1}{2}|a|_{L^3}^3$$

• But now in turn this means that for $\beta > 1$ tbd, we have:

$$\frac{d}{dt}\left(\frac{-E}{F^{\beta}}\right) = \frac{1}{F^{\beta+1}}\left(\beta E \frac{dF}{dt} - F \frac{dE}{dt}\right)$$

$$= \frac{1}{F^{\beta+1}}\Big[-\beta |\partial_y a|_{L^2}^4 + |a|_{L^2}^2 |\partial_y^2 a|_{L^2}^2 + 3\beta |\partial_y a|_{L^2}^2 |a|_{L^3}^3 - \beta \frac{3}{4} |a|_{L^3}^6 + |a|_{L^2}^2 |a|_{L^4}^4 \Big]$$

$$\geq \frac{1}{F^{\beta+1}} \Big[\left(3\beta - \beta/2\right) |a|_{L^3}^3 |\partial_y a|_{L^2}^2 + \left(1 - \frac{3\beta}{4}\right) |a|_{L^3}^6 \Big]$$

≥ 0 when $\beta < 4/3$!

and thus $-E \geq F^{\beta} \left(\frac{-E(a_0)}{F(a_0)^{\beta}} \right)$.

• But now this in turn yields:

$$\frac{d}{dt} F \geq 4 F^{7/6} \left(\frac{-E(a_0)}{F(a_0)^{\beta}} \right)$$

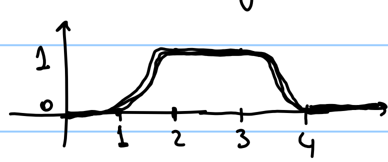
$F(a_0) > 0$

$\Rightarrow$ finite time blowup for $F(a)$!

∎

Remark: $\exists\, a_0$ obeying $F(a_0) > 0$; $E(a_0) < 0$.

Let χ be



a bump function.

Let $a_0(y) = \chi(y/R)$. Then $E(a_0) = \frac{1}{2}\frac{1}{R} |\chi'|_{L^2}^2 - \frac{1}{4} R |\chi|_{L^3}^3$

$\to -\infty$ as $R \to \infty$

∎

Note: At the time of writing of E & Engquist's paper, there were two local existence results known for Prandtl.

$\boxed{\text{Oleinik '66}}$

Assumptions:
$$\begin{cases} U^E > 0 \;;\; U^E \text{ smooth } (C^1_{t,x}) \;;\; \dfrac{\partial_t U^E}{U^E} \in C^0 \cap L^\infty \\ 0 < k_1 \left(U_0^E(x) - u_0^P \right) \leq \partial_Y u_0^P \leq k_2 \left(U_0^E(x) - u_0^P \right) \\ u_0^P \text{ smooth } \quad (C^1_x C^3_Y) \\ + \text{ compatibility conditions} \end{cases}$$

where the spacial domain is $[0,M] \times \mathbb{R}_+$.

Conclusion: Then, $\exists!$ solution u^P of Prandtl, $u^P \in C^1_x C^2_Y$

$$\begin{cases} u^P/_{U^E} \;;\; \partial_Y u^P/_{U^E} \in C^0 \cap L^\infty \\ \exp(-C_1 Y) \leq 1 - \dfrac{u^P}{U^E} \leq \exp(-C_2 Y) \\ 0 < \dfrac{\partial_Y u^P}{U^E} \to 0 \quad \text{as } Y \to \infty. \end{cases}$$

Main idea: "Crocco variables"

$$\begin{matrix} t \mapsto \tau ; \\ x \mapsto \xi ; \end{matrix} \quad Y \mapsto \frac{u^P(x,y,t)}{U^E(x,t)} \;;\; w(\tau,\xi,\eta) = \frac{\partial_y u(x,y,t)}{U^E(x,t)}$$

Then: • domain becomes $\{ 0 < \xi < M \;;\; 0 < \eta < 1 \;,\; 0 < \tau < T \}$

• equation becomes:

$$\partial_\tau w + \eta U^E \partial_\xi w - A \partial_\eta w - Bw = w^2 \partial_{\eta\eta} w$$

where $A = (\eta^2 - 1) \partial_x U^E + (\eta - 1) \dfrac{\partial_t U^E}{U^E}$

$$B = -\eta \partial_x U^E - \frac{\partial_t U^E}{U^E} \quad ; \quad C = \partial_x U^E + \frac{\partial_t U^E}{U^E}$$

- The boundary conditions become:

$$\begin{cases} w\big|_{\eta=1} = 0 \\ w\,\partial_\eta w - v_0 w + c\big|_{\eta=0} = 0 \end{cases}$$

- The point is that this equation is now amenable to MAXIMUM PRINCIPLE type arguments.

Note: E & Engquist I.C. do not obey these conditions.

$$\boxed{\text{Sammartino - Caflisch '98}}$$

Assume: • u_0^P is real analytic w.r.t x & Y, and decays exponentially as $Y\to\infty$
• U_0^E is analytic wrt x.

Then: $\exists\, T_0 > 0$ & $\exists !$ solution u^P on $[0, T_0)$, which is real-analytic wrt. x & Y, and decays exponentially as $Y\to\infty$.

Moreover: Under these assumptions, we have:

$$u^{NS} = (u^E - U^E) + u^P + \underbrace{\sqrt{\nu}\,\tilde{u}^E + \sqrt{\nu}\,\tilde{u}^P}_{\text{linearizations around Euler \& Prandtl}} + \underbrace{w}_{O(\sqrt{\nu})}$$

so that the inviscid limit holds, with rate $\sqrt{\nu}$ on $[0, T_0]$

Main ideas of the proof:
- invert $\partial_t - \partial_Y^2$ and write equations in "mild form" as $\partial_t u = F(u,t)$
- apply Abstract - Cauchy - Kowalewskaya when F is bdd. & Lip. in a scale of Banach spaces X_ρ (ρ = analyticity radius).

Note: E & Engquist's datum cannot be real-analytic wrt y, since they take it to have compact support in y.

- We need a better local existence result!

1

- The initial datum of E-Engquist was analytic wrt x, possibly, but not w.r.t. y.

- Moreover (this is in fact the true motivation), we have that the Prandtl equations are parabolic with respect to (t,y), so that analyticity wrt y should be totally not needed to solve them.

<u>Theorem:</u> (Cannone - Lombardo - Sammartino '03)
(Kukavica - Vicol '11) → (remove conditions on y smoothness and on exponential decay wrt y of the solution)

- Assume that U_0^E is real-analytic wrt x, uniformly (on $\mathbb{R}$).
- Assume that u_0^P is
  - real-analytic wrt x, uniformly (on $\mathbb{R}$)
  - decays at least like $1/Y$ towards U^E as $Y \to \infty$
  - lies in H^1_Y
- Then $\exists!$ local in time solution of the Prandtl equations.

<u>Proof</u> (<u>Sketch</u>)

- For simplicity of the presentation, let's assume we have a constant U_0^E wrt. x, and thus that $\begin{cases} \partial_x P^E = 0 \\ \partial_x U^E = 0 \end{cases}$

- Otherwise, consider

$$\tilde{Y} = Y\, A(x,t)$$

where $A(x,t)$ solves:

$$\begin{cases} \partial_t A + U^E \partial_x A = A\, \partial_x U^E \\ A\big|_{t=0} = 1 \end{cases}$$

and make this change of variables in <u>Prandtl</u>.

- Then, let $\varphi(t,y)$ be the solution of

$$\left.\begin{cases} \partial_t \varphi - \partial_y^2 \varphi = 0 \\ \varphi\big|_{y=0} = 0 \\ \varphi\big|_{y=\infty} = U^E \end{cases}\right\}$$ with initial datum

$\varphi_0 =$ 

$$\varphi_0(y_p) = U^E \operatorname{erf}\left(\tfrac{y}{2}\right)$$

$\Downarrow$

$$\varphi(y,t) = U^E \operatorname{erf}\left(\frac{y}{2\sqrt{t+1}}\right).$$

- Consider instead of the Prandtl equations:

$$\begin{cases} \partial_t u^P - \partial_y^2 u^P + u^P \partial_x u^P + v^P \partial_y u^P = 0 \\ v^P = -\int_0^y \partial_x u^P \\ u^P\big|_{y=0} = 0 \\ u^P\big|_{y=\infty} = U^E \end{cases}$$

③

The equation for

$$\boxed{u = u^P - \varphi}$$

the perturbation around the B.C. lift.

- The equation obeyed by u is:

$$\begin{cases} \partial_t u - \partial_y^2 u + (u+\varphi)\,\partial_x u + v\,\partial_y(u+\varphi) = 0 \\ v = -\int_0^y \partial_x u \\ u|_{y=0} = u|_{y=\infty} = 0 \end{cases}$$

perturbed Prandtl velocity

where we have used that $\partial_x \varphi = 0$!

- Similarly, the equation obeyed by the perturbed vorticity

$$\boxed{\omega = \partial_y u = \partial_y u^P - \partial_y \varphi = \omega^P - \frac{1}{\sqrt{\pi(t+1)}} \exp\left(-\frac{y^2}{4(t+1)}\right)}$$

is given by:

$$\begin{cases} \partial_t \omega - \partial_y^2 \omega + (u+\varphi)\,\partial_x \omega + v\,\partial_y(\omega + \partial_y \varphi) = 0 \\ u = \int_0^y \omega \;, \quad v = -\int_0^y \partial_x u \\ \partial_y \omega|_{y=0} = 0 \\ \omega|_{y=\infty} = 0 \end{cases}$$

perturbed Prandtl vorticity

Real-analytic
Energy estimates

- let $\boxed{\rho(y)}$ be a weight growing at ∞ sufficiently fast, so that $1/\rho^2 \in L^1(0,\infty)$.

  Ex: $\rho(y) = \exp\left(\frac{y^2}{4(t+1)} \right)$ ← this one may depend on t too; good for long time

  $\rho(y) = \langle y \rangle^{1/2+\varepsilon}$ ← this requires the least decay as $y \to \infty$

  $\rho(y) = \exp(\alpha y)$. ← simplest to work with, and this is what we choose, for clarity of the presentation.

- let $\boxed{\tau(t)} > 0$, decreasing be a radius of analyticity.

- Define the norms:

$$\| v \|_{X_\tau} = \sum_{m \geq 0} \| \rho(y) \, \partial_x^m v(x,y) \|_{L^2(\mathbb{H})} \frac{\tau^m (m+1)^2}{m!}$$

$$\| v \|_{Y_\tau} = \sum_{m \geq 1} \| \rho(y) \, \partial_x^m v(x,y) \|_{L^2(\mathbb{H})} \frac{\tau^{m-1} (m+1)^2}{(m-1)!}$$

Note: if $\| v \|_{X_\tau} < \infty \Rightarrow v$ is real-analytic with radius τ w.r.t. x, uniformly in x.

Velocity estimate

$$\frac{d}{dt}\|u\|_{X_\tau} = \dot{\tau}\,\|u\|_{Y_\tau} + \sum_{m\geq 0}\left(\frac{d}{dt}\|\rho\,\partial_x^m u\|_{L^2}\right)\frac{\tau^m\,(m+1)^2}{m!}$$

- $\forall\, m\geq 0$:

$$\frac{1}{2}\frac{d}{dt}\|\rho\,\partial_x^m u\|_{L^2}^2 = \int \rho^2\,\partial_x^m u\ \partial_t(\partial_x^m u)$$

$$= \int \rho^2\,\partial_x^m u\left(\partial_y^2\partial_x^m u - \partial_x^m(u\,\partial_x u + v\,\partial_y u) - \partial_y\varphi\,\partial_x^m v\right)$$

[where we've used that since $\partial_x\rho = \partial_x\varphi = 0$: $\int \rho^2\partial_x^m u\ \partial_x^m(\varphi\,\partial_x u)$]
$= 0$!

$$= -\|\rho\,\partial_y\partial_x^m u\|_{L^2}^2 + \frac{1}{2}\int \partial_y^2(\rho^2)\,(\partial_x^m u)^2$$

$$- \int \rho^2\,\partial_x^m v\ \partial_x^m u\ \partial_y\varphi$$

$$- \int \rho^2\,\partial_x^m u\ \partial_x^m(u\,\partial_x u + v\,\partial_y u)$$

- Since $\rho(y) = \exp(\alpha y) \Rightarrow \frac{1}{2}\partial_y^2(\rho^2) = 2\alpha^2\rho^2$

- Also: $\partial_x^i v(x,y) = -\int_0^y \partial_x^{i+1}u(x,z)\,\rho(z)\ \rho^{-1}(z)\,dz$

$$\Rightarrow \|\partial_x^i v\|_{L_x^2 L_y^\infty} \leq \|\rho\,\partial_x^{i+1}u\|_{L^2_{x,y}}\ \|\rho^{-1}\|_{L^2_y} \leq \frac{C}{\sqrt{\alpha}}\|\rho\,\partial_x^{i+1}u\|_{L^2_{x,y}}$$

- Moreover, since $\rho = \frac{1}{\alpha} \partial_y \rho$

$$\|\rho \partial_x^i u\|_{L^2}^2 = \int \rho^2 (\partial_x^i u)^2 = \frac{1}{\alpha} \int \rho \partial_y \rho \, (\partial_x^i u)^2$$

$$= \frac{1}{2\alpha} \int \partial_y(\rho^2) \, (\partial_x^i u)^2$$

$$= -\frac{1}{2\alpha} \int \rho^2 \, \partial_x^i u \, \partial_y \partial_x^i u$$

$$\leq \frac{1}{2\alpha} \|\rho \partial_x^i u\|_{L^2} \|\rho \partial_y \partial_x^i u\|_{L^2},$$

we have the Poincaré-type inequality

$$2\alpha \|\rho \partial_x^i u\|_{L^2} \leq \|\rho \partial_y \partial_x^i u\|_{L^2}.$$

- Inserting these observations into the H_x^m estimate, we arrive at: (upon <u>dividing</u> by $\|\rho \partial_x^m u\|_{L^2}$)

$$\frac{d}{dt} \|\rho \partial_x^m u\|_{L^2} + 2\alpha \|\rho \partial_y \partial_x^m u\|_{L^2}$$

<u>bdd</u> since $\partial_y \varphi$ decays as $\exp(-y^2/4(t+1))$

$$\leq \alpha^2 \|\rho \partial_x^m u\|_{L^2} + \frac{C}{\sqrt{\alpha}} \|\rho \partial_y \varphi\|_{L^2_y} \|\rho \partial_x^{m+1} u\|_{L^2}$$

$$+ \sum_{j=0}^{m} \binom{m}{j} \|\rho \left(\partial_x^j u \, \partial_x \partial_x^{m-j} u + \partial_x^j v \, \partial_y \partial_x^{m-j} u \right)\|_{L^2}$$

↳ the point here is that we don't even need to integrate by parts in x or y!

• We have: two cases $\begin{cases} 0 \le j \le m/2 \\ m/2 < j \le m \end{cases}$ ← let's only do this case.

$$\| \rho\, \partial_x^j u \; \partial_x^{m-j+1} u \|_{L^2}$$

$$\le \underbrace{\| \rho\, \partial_x^{m-j+1} u \|_{L^2_y L^\infty_x}}_{\substack{\text{by interpolation} \\ |f|_{L^\infty_x} \le |f|^{1/2}_{L^2_x} |\partial_x f|^{1/2}_{L^2_x}}} \quad \underbrace{\| \partial_x^j u \|_{L^\infty_y L^2_x}}_{\substack{\text{using that } \partial_x^j u|_{y=0} = 0 \\ \partial_x^j u = \int_0^y \partial_y \partial_x^j u}}$$

$$\le \| \rho\, \partial_x^{m-j+1} u \|^{1/2}_{L^2} \, \| \rho\, \partial_x^{m-j+2} u \|^{1/2}_{L^2} \qquad \Downarrow \quad \le \| \rho\, \partial_y \partial_x^j u \|_{L^2_{x,y}} \, \| \rho^{-1} \|_{L^2_y}$$

$$\le C_\alpha \, \| \rho\, \partial_x^{m-j+1} u \|^{1/2}_{L^2} \, \| \rho\, \partial_x^{m-j+2} u \|^{1/2}_{L^2} \, \| \rho\, \partial_y \partial_x^j u \|_{L^2}$$

$$\Rightarrow \frac{\tau^m (m+1)^2}{m!} \binom{m}{j} \| \rho\, \partial_x^j u \cdot \partial_x^{m-j+1} u \|_{L^2}$$

$$\le C_\alpha \left(\frac{\| \rho\, \partial_x^{m-j+1} u \|_{L^2} \, \tau^{m-j} (m-j+2)^2}{(m-j)!} \right)^{1/2} \left(\frac{\| \rho\, \partial_x^{m-j+2} u \|_{L^2} \, \tau^{m-j+1} (m-j+3)^2}{(m-j+1)!} \right)^{1/2} \to \|u\|_{Y_\tau}$$

$$\left(\frac{\| \rho\, \partial_y \partial_x^j u \|_{L^2} \, \tau^j (j+1)^2}{j!} \right) \to \| \partial_y u \|_{X_\tau}$$

$$\tau^{-1/2} \, \frac{(m+1)^2 \sqrt{m-j+1}}{(m-j+2)^2 (j+1)^2} \to \le \text{const. since } j \ge m/2$$

A similar bound is done for v!

$$\|\rho\, \partial_x^j v \; \partial_x^{m-j} \partial_y u\|_{L^2}$$

$$\leq \|\partial_x^j v\|_{L^\infty_y L^2_x} \; \|\rho\, \partial_x^{m-j} \partial_y u\|_{L^2_y L^\infty_x}$$

$$\leq \|\rho^{-1}\|_{L^2_y} \; \|\rho\, \partial_x^{j+1} u\|_{L^2_{x,y}} \quad \|\rho\, \partial_x^{m-j} \partial_y u\|_{L^2_{x,y}}^{1/2} \; \|\rho\, \partial_x^{m-j+1} \partial_y u\|_{L^2_{x,y}}^{1/2}$$

$$\Rightarrow \frac{\tau^m (m+1)^2}{m!} \binom{m}{j} \|\rho\, \partial_x^j v \; \partial_x^{m-j} \partial_y u\|_{L^2}$$

$$\leq C_\alpha \underbrace{\left(\frac{\|\rho\, \partial_x^{j+1} u\|_{L^2} \, \tau^j \, (j+2)^2}{j!} \right)}_{\to \|u\|_{Y_\tau}} \left(\frac{\|\rho\, \partial_x^{m-j} \partial_y u\|_{L^2} \, \tau^{m-j} \, (m-j+1)^2}{(m-j)!} \right)^{1/2}$$

$$\left(\frac{\|\rho\, \partial_x^{m-j+1} \partial_y u\|_{L^2} \, \tau^{m-j+1} \, (m-j+2)^2}{(m-j+1)!} \right)^{1/2} \to \|\partial_y u\|_{X_\tau}.$$

$$\cdot\, \tau^{-1/2} \; \frac{(m+1)^2 \sqrt{m-j+1}}{(j+2)^2 \, (m-j+1)^2} \to \leq C \text{ since } j \geq m/2$$

- Using that $\ell^1 * \ell^1 \subseteq \ell^1$, we sum over $m \geq 0$ and obtain:

$$\frac{d}{dt}\|u\|_{X_\tau} + c\,\|\partial_y u\|_{X_\tau}$$

$$\leq c\,\|u\|_{X_\tau}$$

$$+ \|u\|_{Y_\tau}\left(\dot{\tau} + c + c\,\tau^{-1/2}\|\partial_y u\|_{X_\tau}\right)$$

· If we ensure that

$$\dot{\tau} + c + c\,\tau^{-1/2}\,\|\partial_y u\|_{X_\tau} \leq 0$$

then we have

$$\frac{d}{dt}\|u\|_{X_\tau} + c\,\|\partial_y u\|_{X_\tau} \leq c\,\|u\|_{X_\tau}$$

$\Rightarrow$ on $[0, T)$

$$\|u\|_{X_\tau} \leq \|u_0\|_{X_{\tau_0}} \exp(CT)$$

$$\int_0^T \|\partial_y u\|_{X_\tau}\,dt \leq \|u_0\|_{X_{\tau_0}}\left(1 + CT\exp(CT)\right)$$

$\Rightarrow$ may choose:

$$\dot{\tau} + c\,(1+\tau^{-1/2})\,(1+\|\partial_y u\|_{X_\tau}) = 0$$

since in L^1 time

$\Rightarrow \exists\, T_0$:

$\tau(t) \geq T_0/2$

$\forall\, t \in [0, T_0]$

∎

Remark: This proof only gives local existence!

Remark: Same estimates work at level of vorticity perturbation ω.

Thm: Assume additionally that

$$\| \omega_0(x,y) + \frac{y}{2} u_0(x,y) \|_{X_{\tau_0}} \leq \varepsilon$$

for some $\varepsilon \ll 1$, (and Gaussian weights wrt. y.)

where $\frac{1}{\log 1/\varepsilon} \lesssim \tau_0 \lesssim \frac{1}{\varepsilon^3}$.

Then time of existence $\gtrsim \exp\left(\frac{1/\varepsilon}{\log 1/\varepsilon}\right)$.

→ "Almost global existence" → for small datum.

(Ignatova - V. '15).

Q: what if the initial datum is not real-analytic wrt. x, do we still have local existence?

YES: → in two instances → Oleinik (Masmoudi & Wong)
→ Gerard-Varet & Masmoudi

- Main idea of Masmoudi – Wong.

Assume want to close H^s estimate for u^P or ω^P.

$$\Rightarrow \quad \partial_t (\partial_x^s u^P) - \partial_y^2 (\partial_x^s u^P) + u^P \partial_x (\partial_x^s u^P) + \partial_x^s v^P \underbrace{\partial_y u^P}_{= \omega^P} = \text{l.o.t.}$$

$$\partial_t (\partial_x^s \omega^P) - \partial_y^2 (\partial_x^s \omega^P) + u^P \partial_x (\partial_x^s \omega^P) + \partial_x^s v^P \, \partial_y \omega^P = \text{l.o.t.}$$

[all other derivatives are in fact ok, even at the level of H^s ... worst term is from
{no y-derivative on v
{all x-derivatives on v
$\Rightarrow$ $(s+1)$ x-derivatives on u, integrated in y
↳ can't gain from this.]

{Observation: multiply $\partial_x^s u^P$ eqn. by $\dfrac{\partial_y \omega^P}{\omega^P}$

and subtract $\partial_x^s \omega^P$ equation,

the bad terms with $\partial_x^s v^P$ CANCEL ALTOGETHER.

Good energy:
$$\boxed{\|\omega^P\|^2_{\tilde{H}^s} = \sum_{\substack{|\alpha| \le s \\ \alpha \neq (s,0)}} \|\partial^\alpha \omega^P\|^2_{L^2 \text{(weighted in y)}} + \left\| \partial_x^s \omega^P - \frac{\partial_y \omega^P}{\omega^P} \partial_x^s u^P \right\|^2_{L^2}}$$

- combined with maximum principle arguments which guarantee that $\omega^P(x,y) \geq c\,(1+y)^{-\sigma}$ $\forall\, x \in \mathbb{T}^1;\ y \geq 0$, this norm has a good energy estimate, and is equivalent to the H^s-norm.

Q: What happens if ω^P vanishes, i.e. ω^P has critical points, and we are not analytic wrt x? Can one solve locally in time?

Grenier '00 ; Gerard-Varet & Dormy '12 $\to$ linear

non-sharp, ill-posedness in H^s

(Guo-Nguyen '14 ; Gerard-Varet & Nguyen '14 $\to$ non linear)

$$\boxed{\text{Prove that Prandtl is ill-posed in Gevrey-}s \text{ with } s \geq 2.}$$

Main idea: (Gerard-Varet & Dormy).

- Consider a shear flow $(U_S(y), 0)$
  with a non-degenerate critical point: $U_S'(a) = 0$.
- Linearized Prandtl becomes (assuming $U_S(0) = U^E = 0$, $U_S(\infty) =$)

$$\begin{cases} \partial_t \tilde{u} + U_S \, \partial_x \tilde{u} - U_S' \, \tilde{v} - \partial_y^2 \tilde{u} = 0 \\ \tilde{u}\big|_{y=0} = \tilde{u}\big|_{y=\infty} = 0 \end{cases}$$

- High-frequency analysis in x, t: make ansatz:

$$\begin{cases} \tilde{u}(t,y) = e^{ik(\omega(k)t + x)} \, \hat{u}_k(y) \\ \tilde{v}(t,y) = k \, e^{ik(\omega(k)t+x)} \, \hat{v}_k(y). \end{cases}$$

$\to$ since divergence-free: $\boxed{\partial_y \hat{v}_k = -i \hat{u}_k}$.

- Inserting this in the ODE one gets:

$$\begin{cases} \left(\omega(k) + U_S(y)\right) \partial_y \hat{v}_k - \hat{v}_k \, U_S'(y) + i \frac{1}{k} \partial_y^3 \hat{v}_k = 0 \\ \hat{v}_k\big|_{y=0} = \partial_y \hat{v}_k\big|_{y=0} = 0. \end{cases}$$

- They then show that $\exists$ solution to the ODE:

$$\begin{cases} \omega(k) \sim -U_s(a) + \frac{1}{\sqrt{k}}\tau + O(1/k) \\ \hat{v}_k(y) \sim H(y-a)\left(U_s(y) - U_s(a) + \frac{\tau}{\sqrt{k}}\right) + \frac{1}{\sqrt{k}} V\left(\frac{y-a}{k^{1/4}}\right) + O(1/k) \end{cases}$$

for y near a.

internal shear layer.

solutions of the "invisid" equation $k = \infty$!

$\boxed{\text{WHERE} \quad \mathrm{Im}\,\tau < 0}$ $\rightarrow$! HARD

- Upshot:

$$\tilde{v}(t,y) \simeq k \underbrace{e^{i\tau\sqrt{k}}} e^{ikx} \hat{v}_k(y)$$

$$\mathrm{Re}(i\tau) = -\mathrm{Im}\,\tau > 0$$

$\rightarrow$ growth at rate $\sqrt{k}$ $\Rightarrow$ Gevrey 2 !

Thm: (Gerard-Varet & Masmoudi) If u_0^p has non-degenerate critical points, one can solve Prandtl in $G_x^{7/4} H_y^s$.