CHARACTER THEORY OF FINITE GROUPS

Chapter 1:

REPRESENTATIONS

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where $GL(n, K)$ is the group of invertible $n \times n$ matrices over K.

The positive integer n is the degree of $\mathcal{X}$.

K-representations $\mathcal{X}$ and $\mathcal{Y}$ of G, both of degree n, are similar if there exists an invertible $n \times n$ matrix P over K such that

$$\mathcal{Y}(g) = P^{-1}\mathcal{X}(g)P$$

for all $g \in G$.

A K-representation $\mathcal{X}$ of G of degree n is irreducible if there does NOT exist a nonzero proper subspace U of the n-dimensional row space over K such that $U\mathcal{X}(g) \subseteq U$ for all $g \in G$.

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To say that $\mathcal{X}$ is irreducible thus means that it is NOT similar to a representation $\mathcal{Y}$ of the form

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Note: Here, $\mathcal{U}$ and $\mathcal{V}$ are representations.

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If $\mathcal{X}$ is a K-representation of G with degree n, then $\mathcal{X}$ can be extended linearly to $K[G]$.

The result is an algebra homomorphism

$$\mathcal{X} : K[G] \to M_n(K)\,,$$

where $M_n(K)$ is the algebra of $n \times n$ matrices.

Fact: Assuming that K is algebraically closed, the K-representation $\mathcal{X}$ of G is irreducible iff $\mathcal{X}$ maps $K[G]$ onto the full algebra $M_n(K)$.

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Note: The "if" direction is obvious but "only if" requires algebraically closed.

If $\mathcal{X}$ is irreducible and K is algebraically closed, it follows that the only matrices that centralize $\mathcal{X}(G)$ are the scalar matrices. This is called Schur's lemma.

EXERCISE (1.1): Let K be an arbitrary field, and let z be the sum of all elements of G in the group algebra $K[G]$. Show that the one-dimensional subspace Kz of $K[G]$ is an ideal. Also, prove that z is nilpotent iff the characteristic of K divides $|G|$.

EXERCISE (1.2): With the notation as above, show that $K[G]$ has a proper ideal I such that $Kz + I = K[G]$ iff the characteristic of K does not divide $|G|$.

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Note:

$$|G| = \dim(\mathbb{C}[G]) = \sum_{i=1}^{k} (n_i)^2 .$$

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Also, the sums in $\mathbb{C}[G]$ of the elements of each conjugacy class of G form a basis for $\mathbf{Z}(\mathbb{C}[G])$, and thus

$$k = \text{number of classes of } G\,.$$

Observe that the composite map

$$G \to \mathbb{C}[G] \to M_{n_i}(\mathbb{C})$$

is an irreducible $\mathbb{C}$-representation of G.

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Write $\mathcal{X}_i$ to denote this representation.

Fact: Every irreducible $\mathbb{C}$-representation of G is similar to one of the $\mathcal{X}_i$.

Chapter 2:

CHARACTERS

Given a $\mathbb{C}$-representation $\mathcal{X}$ of G, the associated character χ is the function $G \to \mathbb{C}$ given by:

$$\chi(g) = \mathrm{trace}(\mathcal{X}(g)) \, .$$

We say $\mathcal{X}$ affords χ.

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Note: If $\mathcal{X}$ affords χ then $\chi(1)$ is the degree of $\mathcal{X}$. Thus $\chi(1)$ is a positive integer; it is called the degree of χ.

Note:

$$\begin{aligned}\chi(x^{-1}gx) &= \operatorname{trace}(\mathcal{X}(x^{-1}gx)) \\ &= \operatorname{trace}(\mathcal{X}(x)^{-1}\mathcal{X}(g)\mathcal{X}(x)) \\ &= \operatorname{trace}(\mathcal{X}(g)) \\ &= \chi(g)\,.\end{aligned}$$

so characters are constant on conjugacy classes.

The characters of G thus lie in the $\mathbb{C}$-space $\mathrm{cf}(G)$ of class functions of G.

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The characters of G thus lie in the $\mathbb{C}$-space $\mathrm{cf}(G)$ of class functions of G.

Note: We have $\dim(\mathrm{cf}(G)) = k$, the number of classes of G.

If α and β are characters of G afforded by $\mathcal{U}$ and $\mathcal{V}$ respectively, then the representation $\mathcal{X}$ given by

$$\mathcal{X}(g) = \begin{bmatrix} \mathcal{U}(g) & 0 \\ 0 & \mathcal{V}(g) \end{bmatrix}$$

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Thus sums of characters are characters.

A character that is not a sum of two characters is said to be irreducible.

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Notation: $\mathrm{Irr}(G)$ is the set of irreducible characters of G.

Recall: The $\mathbb{C}$-representations $\mathcal{X}_i$ for $1 \leq i \leq k$ come from the decomposition of the group algebra $\mathbb{C}[G]$ as a direct sum of matrix rings.

Also recall: Up to similarity, these $\mathcal{X}_i$ are all of the irreducible $\mathbb{C}$-representations of G.

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THEOREM: *The characters afforded by the* $\mathcal{X}_i$ *are distinct and linearly independent, and they form the set* $\mathrm{Irr}(G)$.

Proof: Since the $\mathcal{X}_i$ come from projections into direct summands of $\mathbb{C}[G]$, we can find $a_j \in \mathbb{C}[G]$ such that $\mathcal{X}_i(a_j) = 0$ if $i \neq j$, but $\mathcal{X}_i(a_i)$ is an arbitrary matrix of the appropriate size.

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Let χ_i be the character afforded by $\mathcal{X}_i$, and view χ_i as being defined on the whole of $\mathbb{C}[G]$. Then we can choose the a_i such that $\chi_i(a_j) = 0$, but $\chi_i(a_i) = 1$. It follows that the χ_i are distinct and linearly independent.

Now let $\chi \in \mathrm{Irr}(G)$ be arbitrary, and let $\mathcal{X}$ be a $\mathbb{C}$-representation affording χ. We argue that $\mathcal{X}$ is irreducible.

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Otherwise, $\mathcal{X}$ is similar to a representation $\mathcal{Y}$ such that

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Then $\chi = \alpha + \beta$, where $\mathcal{U}$ afords α and $\mathcal{V}$ affords β. This contradicts the irreducibility of χ.

We now know that each character $\chi \in \mathrm{Irr}(G)$ is afforded by an irreducible representation, and hence χ is afforded by some $\mathcal{X}_i$. It follows that $\chi = \chi_i$ for some i.

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What remains is to show that each character χ_j is irreducible.

Otherwise, since χ_j is a sum of irreducible characters, it is a sum of characters χ_i with $i \neq j$, and this contradicts the linear independence. $\blacksquare$

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COROLLARY: The set $\mathrm{Irr}(G)$ is a basis for the space $\mathrm{cf}(G)$ of class functions.

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The identity is 1_G and $\lambda^{-1} = \overline{\lambda}$, the complex conjugate of λ.

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The linear characters of G, therefore, are exactly the linear characters of the abelian group G/G', and the number of these is $|G/G'|$. ∎

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Fact: The group of linear characters of G is isomorphic to G/G'.

EXERCISE (2.1): Compute the degrees of the irreducible characters of S_3, A_4, S_4 and A_5.

EXERCISE (2.2) : If χ is a character of G and λ is a linear character, define $\lambda\chi$ to be the function defined by $(\lambda\chi)(g) = \lambda(g)\chi(g)$. Show that $\lambda\chi$ is a character, and that it is irreducible iff χ is irreducible.

In fact, if χ and ψ are arbitrary characters of G then the function $\chi\psi$ defined by $(\chi\psi)(g) = \chi(g)\psi(g)$ is a character. We sketch a proof.

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Let A be an $m \times m$ matrix with entries $a_{i,j}$, and let B be an $n \times n$ matrix with entries $b_{k,l}$. The Kronecker product of A and B, denoted $A \otimes B$, is an $mn \times mn$ matrix, defined as follows.

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The rows and columns of $A \otimes B$ are indexed by the set of ordered pairs (u, v) with $1 \leq u \leq m$ and $1 \leq v \leq n$. The $((i, k), (j, l))$-entry of $A \otimes B$ is $a_{i,j}b_{k,l}$.

Note: To write $A \otimes B$ explicitly as a matrix, one must define some specific order on the index set. The choice of order is essentially irrelevant, however.

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It is easy to compute that

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It is easy to compute that

$$\operatorname{trace}(A \otimes B) = \operatorname{trace}(A)\operatorname{trace}(B)\,.$$

Also, if C and D are $m \times m$ and $n \times n$ matrices, respectively, it follows from the definition of matrix multiplication that

$$(A \otimes B)(C \otimes D) = AC \otimes BD\,.$$

Now let $\mathcal{X}$ and $\mathcal{Y}$ be representations of G, affording characters χ and ψ, respectively. Define $\mathcal{Z}$ by $\mathcal{Z}(g) = \mathcal{X}(g) \otimes \mathcal{Y}(g)$.

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It follows from the above remarks that $\mathcal{Z}$ is a representation, and that the character afforded by $\mathcal{Z}$ is the product $\chi\psi$. This shows that products of characters are characters.

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It follows from the above remarks that $\mathcal{Z}$ is a representation, and that the character afforded by $\mathcal{Z}$ is the product $\chi\psi$. This shows that products of characters are characters.

Note: If either χ or ψ is reducible, it is easy to see that $\chi\psi$ is reducible. Even if χ and ψ are irreducible, however, their product is usually not irreducible.

EXERCISE (2.3): Suppose χ is the unique member of $\mathrm{Irr}(G)$ having degree d for some integer d. Show that $\chi(x) = 0$ for all elements $x \in G - G'$.

EXERCISE (2.4): Let χ be a character of G, and let $\mathcal{X}$ afford χ. Define the function λ on G by $\lambda(g) = \det(\mathcal{X}(g))$. Show that λ is a linear character of G and that it does not depend on the choice of the representation $\mathcal{X}$ affording χ.

Notation: $\lambda = \det(\chi)$. Also, the **determinantal order** $o(\chi)$ is the order of $\det(\chi)$ in the group of linear characters of G.

Chapter 3:

CHARACTER VALUES

Let $\mathcal{X}$ be a representation of G with degree d and let $g \in G$ have order n. Then $\mathcal{X}(g)^n = I$, the $d \times d$ identity matrix.

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It follows by linear algebra that the matrix $\mathcal{X}(g)$ is similar to a diagonal matrix whose diagonal entries are n th roots of unity.

If $\mathcal{X}$ affords the character χ, then

$$\chi(g) = \text{trace}(\mathcal{X}(g)) = \epsilon_1 + \cdots + \epsilon_d\,,$$

where the ϵ_i for $1 \leq i \leq d$ are n th roots of unity.

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$$\chi(g) = \text{trace}(\mathcal{X}(g)) = \epsilon_1 + \cdots + \epsilon_d\,,$$

where the ϵ_i for $1 \leq i \leq d$ are n th roots of unity.

Also

$$\chi(g^{-1}) = \text{trace}(\mathcal{X}(g)^{-1}) = \overline{\epsilon}_1 + \cdots + \overline{\epsilon}_d = \overline{\chi(g)}\,,$$

where the overbar is complex conjugation.

THEOREM: *Let χ be a character of a group G, where χ is afforded by a representation $\mathcal{X}$, and let $g \in G$. Then:*

(a) $|\chi(g)| \leq \chi(1)$.

(b) $|\chi(g)| = \chi(1)$ *iff* $\mathcal{X}(g)$ *is a scalar matrix.*

(c) $\chi(g) = \chi(1)$ *iff* $\mathcal{X}(g)$ *is the identity matrix.*

COROLLARY: *Let* χ *be a character of* G. *Then* $\{g \in G \mid \chi(g) = \chi(1)\}$ *is the kernel of every representation affording* χ. *In particular, this subset is a normal subgroup of* G.

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Notation: We write

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}.$$

This normal subgroup is called the kernel of χ, and χ is faithful if $\ker(\chi) = 1$.

EXERCISE (3.1): Let $N \lhd G$ and $\psi \in \mathrm{Irr}(G/N)$. Define $\chi : G \to \mathbb{C}$ by setting $\chi(g) = \psi(Ng)$. Show that $\chi \in \mathrm{Irr}(G)$ and that $N \subseteq \ker(\chi)$. Also, show that the map $\psi \mapsto \chi$ from $\mathrm{Irr}(G/N)$ into $\mathrm{Irr}(G)$ defines a bijection from $\mathrm{Irr}(G/N)$ to the set $\{\chi \in \mathrm{Irr}(G) \mid N \subseteq \ker(\chi)\}$.

Note: It is customary to identify ψ with χ, so we usually pretend that

$$\mathrm{Irr}(G/N) = \{\chi \in \mathrm{Irr}(G) \mid N \subseteq \ker(\chi)\}\,.$$

EXERCISE (3.2): Show that

$$\bigcap \{\ker(\chi) \mid \chi \in \mathrm{Irr}(G)\} = 1\,,$$

the trivial subgroup of G.

EXERCISE (3.3): Let $N \triangleleft G$. Show that

$$N = \bigcap \{\ker(\chi) \mid N \subseteq \ker(\chi)\}\,.$$

Thus, knowing the irreducible characters of G determines the set of all normal subgroups of G.

COROLLARY: *Let* χ *be a character of* G *afforded by* $\mathcal{X}$*. Then* $\{g \in G \mid |\chi(g)| = \chi(1)\}$ *is the preimage in* G *of the normal subgroup of* $\mathcal{X}(G)$ *consisting of scalar matrices. In particular, this set is a normal subgroup of* G*.*

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Notation: We write

$$\mathbf{Z}(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}.$$

This normal subgroup is the center of χ.

EXERCISE (3.4): Let χ be a character of G. Prove that $\mathbf{Z}(\chi)/\ker(\chi)$ is a cyclic subgroup of $\mathbf{Z}(G/\ker(\chi))$.

EXERCISE (3.5): If $\chi \in \mathrm{Irr}(G)$, show that

$$\mathbf{Z}(\chi)/\ker(\chi) = \mathbf{Z}(G/\ker(\chi)) .$$

EXERCISE (3.6): Let P be a p-group for some prime p. Show that P has a faithful irreducible character if and only if $\mathbf{Z}(P)$ is cyclic.

EXERCISE (3.7): Let $\chi \in \mathrm{Irr}(G)$, where G is a simple group of even order. Show that $\chi(1) \neq 2$.

HINT: Let $g \in G$ have order 2. Consider the values of $\chi(g)$ and $\det(\chi)(g)$.

EXERCISE (3.8): Let G be a simple group and suppose $\chi \in \mathrm{Irr}(G)$ has degree 3. Compute $\chi(g)$ for an element $g \in G$ of order 2.

EXERCISE (3.9): Repeat the above if χ has degree 4.

EXERCISE (3.10): If $H \subseteq G$ and χ is a character of G, it is easy to see that the restriction of χ to H, denoted χ_H, is a character of H. Show that $H \subseteq \mathbf{Z}(\chi)$ iff χ_H has the form $e\lambda$, where e is a positive integer and λ is a linear character of H.

EXERCISE (3.11): Let $H \subseteq G$ and let χ be a character of G such that the restriction χ_H is irreducible. Show that $\mathbf{C}_G(H) \subseteq \mathbf{Z}(\chi)$.

Chapter 4:

ORTHOGONALITY

G acts by right multiplication on $\mathbb{C}[G]$, so each element $g \in G$ determines a linear transformation on $\mathbb{C}[G]$.

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We get a $\mathbb{C}$-representation of G by choosing a basis for $\mathbb{C}[G]$.

The character afforded by this representation is the regular character of G, denoted ρ or ρ_G.

To compute ρ, we can use the basis G for $\mathbb{C}[G]$.

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If $\mathcal{R}$ is the corresponding representation, $\mathcal{R}(g)$ is an $|G| \times |G|$ matrix with rows and columns indexed by the elements of G.

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If $g \neq 1$, therefore, all diagonal entries of $\mathcal{R}(g)$ are 0.

We thus have

$$\rho(g) = \begin{cases} 0 & \text{if } g \neq 1 \\ |G| & \text{if } g = 1\,. \end{cases}$$

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Next, we wish to express the character ρ in terms of $\mathrm{Irr}(G)$.

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$$\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi\,.$$

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As a check (but not a proof) observe that

$$|G| = \rho(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \,.$$

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Note: $\sum_{\chi \in \mathrm{Irr}(G)} e_\chi = 1.$

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For arbitrary $u \in \mathbb{C}[G]$, we have

$$\mathcal{X}_\chi(ue_\psi) = \mathcal{X}_\chi(u)\mathcal{X}_\chi(e_\psi) = \begin{cases} \mathcal{X}_\chi(u) & \text{if } \chi = \psi \\ 0 & \text{if } \chi \neq \psi\,. \end{cases}$$

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$$|G| a_g = \rho(g^{-1} e_\chi) = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1) \psi(g^{-1} e_\chi)$$
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$$\begin{aligned}
\frac{\chi(1)^2}{|G|} &= \sum_{g\in G} (a_{g^{-1}})(a_g) \\
&= \sum_{g\in G} \left(\frac{\chi(1)\chi(g)}{|G|}\right)\left(\frac{\chi(1)\overline{\chi(g)}}{|G|}\right) \\
&= \frac{\chi(1)^2}{|G|^2} \sum_{g\in G} \chi(g)\overline{\chi(g)}.
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Thus $1 = \dfrac{1}{|G|}\displaystyle\sum_{g\in G}\chi(g)\overline{\chi(g)} = \dfrac{1}{|G|}\displaystyle\sum_{g\in G}|\chi(g)|^2$.

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Thus $\sum_{g \in G} \chi(g)\overline{\psi(g)} = 0$.

We now have the first orthogonality relation:

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)} = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{if } \chi \neq \psi\,, \end{cases}$$

for $\chi, \psi \in \mathrm{Irr}(G)$.

Notation: For class functions α and β on G, we write

$$[\alpha, \beta] = \frac{1}{|G|} \sum_{g \in G} \alpha(x) \overline{\beta(x)} .$$

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Also $[\beta, \alpha] = \overline{[\alpha, \beta]}$ and $[\alpha\beta, \gamma] = [\alpha, \overline{\beta}\gamma]$.

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Of course, this is *not* a proof.

COROLLARY: *Let* α *be a nonzero class function on* G. *Then* α *is a character iff* $[\alpha, \chi]$ *is a nonnegative integer for all* $\chi \in \mathrm{Irr}(G)$.

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COROLLARY: *Let* χ *be a character of* G. *Then* χ *is irreducible iff* $[\chi, \chi] = 1$.

EXERCISE (4.1): Let $H \subseteq G$ and $\chi \in \mathrm{Irr}(G)$, and recall that χ_H is the character of H obtained by restricting χ to H. Show that $[\chi_H, \chi_H] \leq |G : H|$ with equality iff χ has the value 0 on all elements of $G - H$.

EXERCISE (4.2): If G is abelian, show that $[\chi, \chi] \geq \chi(1)$ for all (not necessarily irreducible) characters χ of G.

EXERCISE (4.3): Let $H \subseteq G$, with H abelian, and let $\chi \in \mathrm{Irr}(G)$. Show that $\chi(1) \leq |G : H|$.

EXERCISE (4.4): Let $\chi \in \mathrm{Irr}(G)$, where G is a nontrivial group. Show that there is a nonidentity element $g \in G$ such that $\chi(g) \neq 0$.

EXERCISE (4.5): Let $H < G$ with H is abelian. If $\chi \in \mathrm{Irr}(G)$ with $|G : H| = \chi(1)$, show that H contains a nonidentity normal subgroup of G.

EXERCISE (4.6): Let G act on a set Ω. Let π be the corresponding permutation character, so $\pi(g) = |\{t \in \Omega \mid t{\cdot}g = t\}|$. Show that π is a character of G and $[\pi, 1_G]$ is the number of orbits of G on Ω.

EXERCISE (4.7): Let $G = A_5$, the alternating group, and let π be the permutation character of the natural action of G on five points. Show that $\chi = \pi - 1_G$ is an irreducible character of G having degree 4.

EXERCISE (4.8): Again let $G = A_5$ and observe that G acts by conjugation on the set of six Sylow 5-subgroups of G. Let σ be the permutation character of this action. Show that $\psi = \sigma - 1_G$ is an irreducible character of G having degree 5.

It is customary to display the irreducible characters of a group G in a character table.

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This is a square array of complex numbers with rows indexed by $\mathrm{Irr}(G)$ and columns indexed by the set of classes of G. The entry in a given row and column is the common value of the given character on the elements of the given class.

Usually, the first column of the character table corresponds to the class of the identity in G, and thus we see the irreducible character degrees of G by reading down this column.

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It is also customary for the first row of the character table to correspond to the principal character 1_G, and thus the first row consists of 1s.

EXERCISE (4.9): Show that knowing the character table of G determines whether or not G is solvable or G is nilpotent.

We introduce some temporary notation now.

Write $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_k\}$ and let g_j be a representative of the j th conjugacy class of G.

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The character table of G is thus the matrix X with (i, j)-entry equal to $\chi_i(g_j)$.

Write d_t to denote the size of the t th conjugacy class of G. Then by the first orthogonality relation, we have:

$$\delta_{i,j} = [\chi_i, \chi_j] = \frac{1}{|G|} \sum_{t=1}^{k} \chi_i(g_t) d_t \overline{\chi_j(g_t)}\,.$$

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We can rewrite this in matrix notation:

$$I = \frac{1}{|G|} X D \overline{X^t} \, ,$$

where $D = \text{diag}(d_1, d_2, \ldots, d_k)$.

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Now $|G|/d_j = |\mathbf{C}_G(g_j)|$, and thus for $x, y \in G$, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(x)}\chi(y) = \begin{cases} |\mathbf{C}_G(x)| & \text{if } x \sim y \\ 0 & \text{otherwise,} \end{cases}$$

where $x \sim y$ means that x and y are conjugate.

This is the second orthogonality relation, also called “column orthogonality”. Since it is a consequence of the first orthogonality relation, or “row orthogonality”, it gives no additional information about the characters of G. It is, however, useful, especially when constructing character tables.

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A consequence is that no two columns of the character table can be identical.

If $x, y \in G$ are not conjugate, therefore, there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) \neq \chi(y)$.

Chapter 5:

INTEGRALITY

An algebraic integer is a complex number that is a root of a monic polynomial in $\mathbb{Z}[x]$.

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Fact: The algebraic integers are a subring of $\mathbb{C}$.

COROLLARY: *Character values are algebraic integers.*

Fact: Rational algebraic integers are in $\mathbb{Z}$.

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Fact: Suppose $u_i \in \mathbb{C}$ for $1 \leq i \leq n$, and let R be the set of $\mathbb{Z}$-linear combinations if the u_i. If R is closed under multiplication, then all u_i are algebraic integers.

Now let $\chi \in \mathrm{Irr}(G)$ and let $\mathcal{X}$ afford χ.

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Let $z \in \mathbf{Z}(\mathbb{C}[G])$ so $\mathcal{X}(z)$ commutes with all matrices in $\mathcal{X}(\mathbb{C}[G])$. Since this is a full matrix ring, we see that $\mathcal{X}(z)$ is a scalar matrix.

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Thus $\mathcal{X}(z) = \omega I$ for some complex number ω, and we have

$$\chi(z) = \mathrm{trace}(\mathcal{X}(z)) = \mathrm{trace}(\omega I) = \omega\chi(1)\,.$$

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It follows that ω depends on χ but not on $\mathcal{X}$, and we write ω_χ for ω.

We have $\omega_\chi(z) = \chi(z)/\chi(1)$.

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Also $\mathcal{X}(z) = \omega_\chi(z)I$, and it follows that ω_χ is an algebra homomorphism $\mathbf{Z}(\mathbb{C}[G]) \to \mathbb{C}$.

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Recall that the conjugacy class sums in $\mathbb{C}[G]$ form a basis for $\mathbf{Z}(\mathbb{C}[G])$ so the central characters are determined by their values on class sums.

Let K be a class of G and write $\widehat{K}$ to denote the sum of the elements of K, so $\widehat{K} \in \mathbf{Z}(\mathbb{C}[G])$.

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Then

$$\omega_\chi(\widehat{K}) = \frac{\chi(\widehat{K})}{\chi(1)} = \frac{|K|\chi(g)}{\chi(1)} .$$

Now suppose K and L are classes of G. Then $\widehat{K}\widehat{L}$ is a linear combination of class sums, so we can write

$$\widehat{K}\widehat{L} = \sum_{M} a_{K,L,M} \widehat{M}\,,$$

where M runs over the classes of G.

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where M runs over the classes of G.

The coefficient of each group element on the left of the above equation is a nonnegative integer, so the coefficients $a_{K,L,M}$ must also be nonnegative integers.

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THEOREM: *Let $g \in G$ and $\chi \in \mathrm{Irr}(G)$, and let K be the class of g in G. Then $\omega_\chi(\widehat{K}) = \dfrac{|K|\chi(g)}{\chi(1)}$ is an algebraic integer.*

THEOREM: $\chi(1)$ *divides* $|G|$ *for all* $\chi \in \mathrm{Irr}(G)$.

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Proof: For each class K of G, let g_K be an element of K. By the first orthogonality relation

$$|G| = \sum_{g \in G} \chi(g)\overline{\chi(g)} = \sum_{K} |K|\chi(g_K)\overline{\chi(g_K)},$$

where the second sum runs over the classes of G.

Since $|K|\chi(g_K) = \omega_\chi(\widehat{K})\chi(1)$, we have

$$|G| = \chi(1) \sum_K \omega_\chi(\widehat{K}) \overline{\chi(x_K)}\,.$$

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Note: A related argument can be used to prove the following stronger result.

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Note: A related argument can be used to prove the following stronger result.

Fact: Let $\chi \in \mathrm{Irr}(G)$. Then $|G : \mathbf{Z}(G)|$ is divisible by $\chi(1)$.

EXERCISE (5.1): Let G be a p-group, where $G' = \mathbf{Z}(G)$ has order p. Prove the following.

(a) Each noncentral class of G has size p.

(b) G has exactly $p-1$ nonlinear irreducible characters.

(c) The average of $\chi(1)^2$ for nonlinear $\chi \in \mathrm{Irr}(G)$ is $|G|/p$.

(d) $\chi(1)^2 \leq |G|/p$ for all $\chi \in \mathrm{Irr}(G)$, and thus $\chi(1)^2 = |G|/p$ for nonlinear $\chi \in \mathrm{Irr}(G)$.

Note: Groups of this type are extraspecial.

We begin work now toward a proof of Burnside's famous $p^a q^b$-theorem.

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LEMMA: *Let χ be a character of G, where $|G| = n$, and let $g \in G$. Suppose $|\chi(g)| < \chi(1)$. Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, where $\mathbb{Q}_n$ is the n th cyclotomic field. Then $|\chi(g)^\sigma| < \chi(1)$ for all $\sigma \in \Gamma$.*

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Proof: We know that $\chi(g)$ is a sum of $\chi(1)$ roots of unity in $\mathbb{Q}_n$. These are not all equal since $|\chi(g)| < \chi(1)$. It follows that $\chi(g)^\sigma$ is also a sum of $\chi(1)$ roots of unity that are not all equal. The result follows. ∎

THEOREM (Burnside): *Let* $\chi \in \mathrm{Irr}(G)$ *and* $g \in G$, *and assume that* $\chi(1)$ *is relatively prime to the size of the class* K *containing* g. *Then either* $\chi(g) = 0$ *or* $g \in \mathbf{Z}(\chi)$.

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Then

$$\frac{\chi(g)}{\chi(1)} = \frac{\chi(g)}{\chi(1)}\big(a\chi(1) + b|K|\big) = a\chi(g) + b\omega$$

is an algebraic integer.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, where $n = |G|$, and let

$$z = \prod_{\sigma \in \Gamma} \left(\frac{\chi(g)}{\chi(1)} \right)^{\sigma},$$

so z is an algebraic integer in $\mathbb{Q}$. Then $z \in \mathbb{Z}$.

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Assume now that $g \notin \mathbf{Z}(\chi)$. Then $|\chi(g)| < \chi(1)$, so by the lemma, $|\chi(g)^{\sigma}| < \chi(1)$ for all $\sigma \in \Gamma$.

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Each factor of z has absolute value less than 1, so $|z| < 1$. Since $z \in \mathbb{Z}$, we have $z = 0$. Then $\chi(g)^{\sigma} = 0$ for some σ, so $\chi(g) = 0$. ∎

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If $\chi \in \mathrm{Irr}(G)$, then $\mathbf{Z}(\chi) \lhd G$. If $\mathbf{Z}(\chi) = G$, then χ is linear. But $G' = G$, so χ is principal.

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If $\chi \in \mathrm{Irr}(G)$, then $\mathbf{Z}(\chi) \lhd G$. If $\mathbf{Z}(\chi) = G$, then χ is linear. But $G' = G$, so χ is principal.

Thus if $\chi \in \mathrm{Irr}(G)$ is nonprincipal, $\mathbf{Z}(\chi) = 1$.

By the previous theorem, $\chi(g) = 0$ for all non-principal χ where p does not divide $\chi(1)$.

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We have

$$0 = \rho(g) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi(g)$$

Separate $\chi = 1_G$ and sum only over χ for which p divides $\chi(1)$.

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Separate $\chi = 1_G$ and sum only over χ for which p divides $\chi(1)$.

This yields $0 = 1 + p\alpha$, where α is an algebraic integer. Since $\alpha = -1/p \in \mathbb{Q}$, this is a contradiction. $\blacksquare$

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Let $P > 1$ be a Sylow subgroup of G. Choose $z \in \mathbf{Z}(P)$ with $z \neq 1$. Let K be the class of z in G, so $|K| = |G : \mathbf{C}_G(z)|$, and this divides $|G : P|$, which is a prime power, by hypothesis. This contradicts the previous theorem. ∎

EXERCISE (5.2): Suppose that no prime other than p or q divides the degree of any member of $\mathrm{Irr}(G)$. Show that G cannot be a nonabelian simple group.

Chapter 6:

INDUCED CHARACTERS

Let $H \subseteq G$. If χ is a character of G then the restriction χ_H is a character of H. In this chapter, we start with a character of H and produce a character of G.

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More generally, start with a class function α of H. We describe how to build the induced class function α^G of G.

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More generally, start with a class function α of H. We describe how to build the induced class function α^G of G.

For $g \in G$, we first define

$$\alpha^0(g) = \begin{cases} 0 & \text{if } g \notin H \\ \alpha(g) & \text{if } g \in H\,. \end{cases}$$

To convert α^0 into a class-function on G we want to sum over all conjugates of g that happen to lie in H.

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For $g \in G$, we define

$$\alpha^G(g) = \frac{1}{|H|} \sum_{x \in G} \alpha^0(xgx^{-1}).$$

Of course, the "normalization factor" $1/|H|$ is not needed to make this a class function, but it is convenient for other reasons.

Note: If g is not conjugate to any element of H, then $\alpha^G(g) = 0$.

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Note: If $H = G$, then

$$\alpha^G(g) = \frac{1}{|G|} \sum_{x \in G} \alpha(xgx^{-1}) = \alpha(g),$$

so $\alpha^G = \alpha$ in this case.

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so $\alpha^G = \alpha$ in this case.

Note: We have

$$\alpha^G(1) = \frac{1}{|H|} \sum_{x \in G} \alpha^0(1) = |G : H|\alpha(1).$$

Note: Let α be the principal character of the trivial subgroup of G. Then by the previous notes, α^G has the value 0 on nonidentity elements of G and $\alpha^G(1) = |G|$. Thus, $\alpha^G = \rho$,

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$$\alpha^G(g) = \sum_{t \in T} \alpha^0(tgt^{-1}),$$

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where T is a set of representatives for the right cosets of H in G.

This works because $\alpha^0((ht)g(ht)^{-1}) = \alpha^0(tgt^{-1})$ for $h \in H$.

EXERCISE (6.1): Suppose $G = HK$, where H and K are subgroups, and let $D = H \cap K$. If α is a class function on H. Show that

$$(\alpha^G)_K = (\alpha_D)^K .$$

EXERCISE (6.2): Let $H \subseteq G$. Let α be a class function on H and let β be a class function on G. Show that

$$(\beta_H \alpha)^G = \beta \alpha^G .$$

EXERCISE (6.3): Let $H \subseteq K \subseteq G$ and let α be a class function on H. Show that

$$(\alpha^K)^G = \alpha^G .$$

THEOREM (Frobenius reciprocity). *Let $H \subseteq G$. Let α be a class function on H and let β be a class function on G. Then*

$$[\alpha^G, \beta] = [\alpha, \beta_H]\,.$$

Proof:

$$[\alpha^G, \beta] = \frac{1}{|G|} \sum_{g \in G} \alpha^G(g) \overline{\beta(g)}$$

$$= \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \alpha^0(xgx^{-1}) \overline{\beta(g)} \, .$$

Proof:

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Change variables: $h = xgx^{-1}$, so $g = x^{-1}hx$.

$$[\alpha^G, \beta] = \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{h \in H} \alpha(h) \overline{\beta(x^{-1}hx)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \alpha(h) \overline{\beta(h)} = [\alpha, \beta_H].$$

COROLLARY: *Let* $H \subseteq G$ *and let* ψ *be a character of* H. *Then* ψ^G *is a character of* G.

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We have $[\psi^G, \chi] = [\psi, \chi_H]$, and since this is the inner product of two characters of H, it is a nonnegative integer. ∎

Note: Let $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G)$, where $H \subseteq G$. Then ψ is a constituent of χ_H iff χ is a constituent of ψ^G.

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Notation: Let $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$. We write $\mathrm{Irr}(G|\psi)$ to denote the set of irreducible characters of G that lie over ψ.

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Notation: Let $H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$. We write $\mathrm{Irr}(G|\psi)$ to denote the set of irreducible characters of G that lie over ψ.

Equivalently, $\mathrm{Irr}(G|\psi)$ is the set of irreducible constituents of ψ^G.

EXERCISE(6.4): Suppose that every nonlinear irreducible character of G has degree at least n. Show that if $H \subseteq G$ and $|G : H| \leq n$, then $G' \subseteq H$, and thus $H \lhd G$.

HINT: Consider the character $(1_H)^G$, where 1_H is the principal character of H.

THEOREM (Frobenius): *Let $H \subseteq G$, and assume $H \cap H^x = 1$ for all $x \in G - H$. Then $N = (G - \bigcup_{x \in G} H^x) \cup \{1\}$ is a subgroup of G.*

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Note: Since $H = \mathbf{N}_G(H)$, the number of distinct conjugates of H is $|G : H|$, and thus

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Note: $N \cap H = 1$, so assuming that N is a subgroup, we have $|NH| = |N||H| = |G|$, and thus $NH = G$.

If H is as in the statement of Frobenius' theorem, and $1 < H < G$, we say that G is a Frobenius group. Also, H is a Frobenius complement in G and the subgroup N is a Frobenius kernel.

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Proof of Frobenius' theorem: Let $\psi \in \mathrm{Irr}(H)$ with $\psi \neq 1_H$, and let $\alpha = \psi - \psi(1)1_H$, so α is a class function on H and $\alpha(1) = 0$. Then $\alpha^G(1) = 0$.

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Let $h \in H$ with $h \neq 1$. Then $x^{-1}hx$ lies in H iff $h \in H^x$, and since $H \cap H^x = 1$ if x is not in H, this happens iff $x \in H$.

We have

$$\alpha^G(h) = \frac{1}{|H|}\sum_{x\in G}\alpha^0(xhx^{-1})$$
$$= \frac{1}{|H|}\sum_{x\in H}\alpha(h) = \alpha(h)\,,$$

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Recall that $\alpha = \psi - \psi(1)1_H$, where $\psi \in \mathrm{Irr}(H)$.

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and thus $(\alpha^G)_H = \alpha$.

Recall that $\alpha = \psi - \psi(1)1_H$, where $\psi \in \mathrm{Irr}(H)$.

Then $[\alpha^G, \alpha^G] = [(\alpha^G)_H, \alpha] = [\alpha, \alpha] = 1+\psi(1)^2$.

We have $[\alpha^G, 1_G] = [\alpha, 1_H] = -\psi(1)$, so we can write $\alpha^G = \Xi - \psi(1)1_G$, where Ξ is a $\mathbb{Z}$-linear combination of nonprincipal irreducible characters of G.

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Then $[\Xi, \Xi] = 1$, so Ξ is plus-or-minus an irreducible character.

Also,

$$\psi - \psi(1)1_H = \alpha = (\alpha^G)_H = \Xi_H - \psi(1)1_H \,,$$

so $\Xi_H = \psi$, and hence $\Xi \in \mathrm{Irr}(G)$.

We now know that for each nonprincipal character $\psi \in \mathrm{Irr}(H)$, there is a character $\Xi_\psi \in \mathrm{Irr}(G)$ such that $(\Xi_\psi)_H = \psi$.

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Let $M = \bigcap_\psi \ker(\Xi_\psi)$. We argue that $M = N$.

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Let $M = \bigcap_\psi \ker(\Xi_\psi)$. We argue that $M = N$.

If $h \in M \cap H$, then $\psi(h) = \Xi(h) = \Xi(1) = \psi(1)$, so $h \in \bigcap\{\ker(\psi) \mid \psi \in \mathrm{Irr}(H)\} = 1$.

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Let $M = \bigcap_\psi \ker(\Xi_\psi)$. We argue that $M = N$.

If $h \in M \cap H$, then $\psi(h) = \Xi(h) = \Xi(1) = \psi(1)$, so $h \in \bigcap\{\ker(\psi) \mid \psi \in \mathrm{Irr}(H)\} = 1$.

Thus $M \cap H = 1$, and hence $M \cap H^x = 1$ for all $x \in G$, and we have $M \subseteq N$.

We must now show that $N \subseteq M$, so let $1 \neq n \in N$. Then n lies in no conjugate of H so $\alpha^G(n) = 0$, where α is as before (depending on a nonprincipal character $\psi \in \mathrm{Irr}(H)$.

We must now show that $N \subseteq M$, so let $1 \neq n \in N$. Then n lies in no conjugate of H so $\alpha^G(n) = 0$, where α is as before (depending on a nonprincipal character $\psi \in \mathrm{Irr}(H)$.
Now

$$0 = \alpha^G(n) = \Xi_\psi(n) - \psi(1) = \Xi_\psi(n) - \Xi_\psi(1)\,,$$

and thus $n \in \ker(\Xi_\psi)$ for all ψ. Then $n \in M$, as required. $\blacksquare$

Alternative statement of Frobenius' theorem:

THEOREM: *Let G be a transitive permutation group on a set Ω, and assume that no nonidentity element of G fixes more than one point of Ω. Then the identity together with the elements of G that fix no point form a subgroup N of G.*

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Note: The subgroup N is normal. Also, it is regular, which means that it is transitive and the stabilizer of a point is trivial.

EXERCISE (6.5): Let $G = NH$, where $N \lhd G$ and $N \cap H = 1$, and assume that $1 < N < G$, Show that H is a Frobenius complement in G iff $\mathbf{C}_N(h) = 1$ for all elements $h \in H$ with $h \neq 1$. Show also that in this case, N is the Frobenius kernel.

EXERCISE (6.6): If H is a Frobenius complement in G and $|H|$ is even, show that the Frobenius kernel is abelian.

Chapter 7:

NORMAL SUBGROUPS

If $\sigma : G \to H$ is an isomorphism of groups, it should be clear that σ carries characters of G to characters of H, and it carries irreducible characters to irreducible characters.

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If χ is a character of G, we write χ^σ to denote the corresponding character of H, where χ^σ is defined by the formula

$$\chi^\sigma(g^\sigma) = \chi(g)\,.$$

If $\sigma : G \to H$ is an isomorphism of groups, it should be clear that σ carries characters of G to characters of H, and it carries irreducible characters to irreducible characters.

If χ is a character of G, we write χ^σ to denote the corresponding character of H, where χ^σ is defined by the formula

$$\chi^\sigma(g^\sigma) = \chi(g) .$$

This also works if $H = G$, so $\sigma \in \mathrm{Aut}(G)$ permutes the members of $\mathrm{Irr}(G)$.

Let $N \triangleleft G$. If $g \in G$, then g induces an automorphism of N, so G acts on the set $\mathrm{Irr}(N)$, and we have

$$\theta^g(n^g) = \theta(n)$$

for $g \in G$ and $n \in N$.

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Equivalently,

$$\theta^g(n) = \theta(gng^{-1}).$$

We say that the characters $\theta^g \in \mathrm{Irr}(N)$ are the conjugates of θ in G.

THEOREM (Clifford): *Let* $\chi \in \mathrm{Irr}(G)$, *and suppose* $N \triangleleft G$. *Then the irreducible constituents of* χ_N *form a* G*-orbit in* $\mathrm{Irr}(N)$. *Also, the multiplicities of each of these constituents in* χ_N *are equal.*

THEOREM (Clifford): *Let $\chi \in \mathrm{Irr}(G)$, and suppose $N \triangleleft G$. Then the irreducible constituents of χ_N form a G-orbit in $\mathrm{Irr}(N)$. Also, the multiplicities of each of these constituents in χ_N are equal.*

In other words,

$$\chi_N = e \sum_{i=1}^{t} \theta_i \, ,$$

where the θ_i are the distinct conjugates of θ in G and e and t are positive integers.

Proof: Let θ be any irreducible constituent of χ_N. If $n \in N$, then for all $x \in G$, we have $xnx^{-1} \in N$. Thus

Proof: Let θ be any irreducible constituent of χ_N. If $n \in N$, then for all $x \in G$, we have $xnx^{-1} \in N$. Thus

$$\theta^G(n) = \frac{1}{|N|} \sum_{x \in G} \theta(xnx^{-1}) = \frac{1}{|N|} \sum_{x \in G} \theta^x(n),$$

so

$$(\theta^G)_N = \frac{1}{|N|} \sum_{x \in G} \theta^x.$$

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so

$$(\theta^G)_N = \frac{1}{|N|} \sum_{x \in G} \theta^x.$$

Since χ is a constituent of θ^G, it follows that χ_N is a sum of characters of the form θ^x for $x \in G$.

Since χ is a class function, we have $\chi^g = \chi$ for $g \in G$, so χ_N is invariant under the G-action.

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Note: The common multiplicity, usually denoted e, is often called the ramification.

COROLLARY: *Let $\chi \in \mathrm{Irr}(G)$, and let θ be an irreducible constituent of χ_N, where $N \triangleleft G$. Then $\chi(1) = et\theta(1)$, so $\theta(1)$ divides $\chi(1)$.*

Notation: $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.

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EXERCISE (7.1): Suppose all members of $\mathrm{cd}(G)$ are powers of the prime p.

(a) If G is nonabelian, show that $|G : G'|$ is divisible by p.

(b) Show that G has an abelian normal subgroup A, where $|G : A|$ is a power of p and p does not divide $|A|$.

Note: The converse is also true. If A exists as in (b) then $\mathrm{cd}(G)$ consists of powers of p. We need a little more theory to prove this, however.

Let $\theta \in \mathrm{Irr}(N)$, where $N \lhd G$, and let $T = G_\theta$ be the stabilizer of θ in the action of G on $\mathrm{Irr}(N)$.

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Note that $N \subseteq T$. The subgroup T is sometimes called the inertia group of θ, and we sometimes write $T = I_G(\theta)$.

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Note that $N \subseteq T$. The subgroup T is sometimes called the inertia group of θ, and we sometimes write $T = I_G(\theta)$.

Recall that in Clifford's theorem, we had

$$\chi_N = e \sum_{i=1}^{t} \theta_i \,.$$

where t is the size of the G-orbit of $\theta = \theta_1$. We thus have $t = |G : T|$, where $T = G_\theta$. In particular, t divides $|G : N|$.

It is also true that e divides $|G : N|$. We will prove this in the case where G/N is solvable.

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THEOREM: *Let $N \triangleleft G$, where $|G : N| = p$, a prime. Then either χ_N is a sum of p distinct irreducible characters or χ_N is irreducible.*

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In the notation of Clifford's theorem, therefore, if $|G : N| = p$ is prime, then $e = 1$ and $t \in \{1, p\}$.

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In the notation of Clifford's theorem, therefore, if $|G : N| = p$ is prime, then $e = 1$ and $t \in \{1, p\}$.

Proof: In general, t divides $|G : N|$, so $t \in \{1, p\}$. Suppose first that $t = p$.

Let θ be an irreducible constituent of χ_N, so $[\chi_N, \theta] = e$, where e is the ramification.

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We have $[\chi, \theta^G] = [\chi_N, \theta] = e$, so χ is a constituent of θ^G with multiplicity e.

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We have $[\chi, \theta^G] = [\chi_N, \theta] = e$, so χ is a constituent of θ^G with multiplicity e.

Since $|G : N| = p = t$, this yields

$$e\chi(1) \leq \theta^G(1) = |G : N|\theta(1) = t\theta(1) \leq \chi(1),$$

so $e = 1$, as required.

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Since $|G : N| = p = t$, this yields

$$e\chi(1) \leq \theta^G(1) = |G : N|\theta(1) = t\theta(1) \leq \chi(1)\,,$$

so $e = 1$, as required.

Now assume $t = 1$, so $\chi_N = e\theta$.

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If $\lambda \in \Lambda$, then

$$(\chi\lambda)_N = \chi_N \lambda_N = \chi_N = e\theta\,,$$

so by Frobenius reciprocity, $\chi\lambda$ is an irreducible constituent of θ^G.

If the Λ-orbit of χ has size p, then θ^G has at least p different irreducible constituents with degree $\chi(1)$, and thus

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$$p\theta(1) = \theta^G(1) \geq p\chi(1) = pe\theta(1)$$

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$$p\theta(1) = \theta^G(1) \geq p\chi(1) = pe\theta(1)$$

so $e = 1$, as wanted.

To complete the proof, we assume that the Λ-orbit of χ has size 1, and we derive a contradiction.

Let $\lambda \in \Lambda$ be nonprincipal. Since $\chi = \chi\lambda$ and $\lambda(x) \neq 1$ for all $x \in G - N$, we have $\chi(x) = 0$ for all such x.

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We have

$$e^2 = [e\theta, e\theta] = [\chi_N, \chi_N] = |G : N| = p\,,$$

where the penultimate equality holds by Exercise 4.1. This is the desired contradiction. $\blacksquare$

COROLLARY: *Let* $N \triangleleft G$, *where* G/N *is solvable, and let* $\chi \in \mathrm{Irr}(G)$. *If* θ *is an irreducible constituent of* χ_N, *then the integer* $\chi(1)/\theta(1)$ *divides* $|G : N|$.

COROLLARY: *Let $N \triangleleft G$, where G/N is solvable, and let $\chi \in \mathrm{Irr}(G)$. If θ is an irreducible constituent of χ_N, then the integer $\chi(1)/\theta(1)$ divides $|G : N|$.*

Proof: In the notation of Clifford's theorem, we have $\chi(1)/\theta(1) = et$, so this is an integer.

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Proof: In the notation of Clifford's theorem, we have $\chi(1)/\theta(1) = et$, so this is an integer.

If $N = G$, then $\chi = \theta$, and the result is trivial. We can thus assume that $N < G$ and we induct on $|G:N|$.

If G/N is simple, then since it is solvable, it has prime order, say p. By the previous theorem, $\chi(1)/\theta(1)$ is 1 or p, so we are done in this case.

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Assuming now that G/N is not simple, there exists $M \triangleleft G$ with $N < M < G$. Let $\psi \in \mathrm{Irr}(M)$ lie under χ and over θ.

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Assuming now that G/N is not simple, there exists $M \triangleleft G$ with $N < M < G$. Let $\psi \in \mathrm{Irr}(M)$ lie under χ and over θ.

By the inductive hypothesis, $\chi(1)/\psi(1)$ divides $|G : M|$ and $\psi(1)/\theta(1)$ divides $|M : N|$.

Thus

$$\frac{\chi(1)}{\theta(1)} = \left(\frac{\chi(1)}{\psi(1)}\right)\left(\frac{\psi(1)}{\theta(1)}\right)$$

divides $|G : M||M : N| = |G : N|$. ∎

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COROLLARY: *Let* $N \lhd G$, *where* G/N *is solvable, and let* $\chi \in \mathrm{Irr}(G)$. *If*

$$\chi_N = e \sum_{i=1}^{t} \theta_i$$

as in Clifford's theorem, then et *divides* $|G : N|$.

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$$\frac{\chi(1)}{\theta(1)} = (\frac{\chi(1)}{\psi(1)})(\frac{\psi(1)}{\theta(1)})$$
divides $|G : M||M : N| = |G : N|$. ∎

COROLLARY: *Let $N \lhd G$, where G/N is solvable, and let $\chi \in \mathrm{Irr}(G)$. If*
$$\chi_N = e \sum_{i=1}^{t} \theta_i$$
as in Clifford's theorem, then et divides $|G : N|$.

Proof: We have $et = \chi(1)/\theta_i(1)$. ∎

COROLLARY: *Let $N \triangleleft G$, where G/N is solvable. Let $\chi \in \mathrm{Irr}(G)$, and suppose that $\chi(1)$ and $|G : N|$ are relatively prime. Then χ_N is irreducible.*

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Proof: Let θ be an irreducible constituent of χ_N. Then $\chi(1)/\theta(1)$ divides both $|G : N|$ and $\chi(1)$. Thus $\chi(1)/\theta(1) = 1$, and so $\theta(1) = \chi(1)$. Since θ is a constituent of χ_N, it follows that $\chi_N = \theta$. ∎

COROLLARY (Itô): *Let* $N \lhd G$, *where* N *is abelian, and assume that* G/N *is solvable. Then the degree of every irreducible character of* G *divides* $|G : N|$.

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Proof: Let $\chi \in \mathrm{Irr}(G)$ and let θ be an irreducible constituent of χ_N. Since N is abelian, $\theta(1) = 1$, so $\chi(1) = \chi(1)/\theta(1)$, and this divides $|G : N|$. ∎

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Fact: All of the preceding corollaries hold even without assuming that G/N is solvable.

EXERCISE (7.2): Show $\mathrm{cd}(G)$ consists of powers of a fixed prime p iff G has an abelian normal subgroup with index a power of p.

HINT: Exercise 7.1.

Note: This is one of many theorems determining the structure of G from the set $\mathrm{cd}(G)$.

EXERCISE (7.3): Let N be a normal Hall π-subgroup of G, where π is a set of primes, and let $\chi \in \mathrm{Irr}(G)$. Show that the degrees of the irreducible constituents of χ_N are equal to the π-part of $\chi(1)$.

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THEOREM (Gallagher correspondence): *Given* $N \lhd G$ *and* $\chi \in \mathrm{Irr}(G)$, *write* $\theta = \chi_N$ *and assume that* θ *is irreducible. Then the map* $\beta \mapsto \beta\chi$ *is a bijection from* $\mathrm{Irr}(G/N)$ *onto* $\mathrm{Irr}(G|\theta)$.

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Note: In order for $\beta\chi$ to make sense for a character $\beta \in \mathrm{Irr}(G/N)$, we view β as a character of G with $N \subseteq \ker(\beta)$.

Proof: Since θ is the restriction to N of a character of G, we see that θ is invariant in G, and it follows that $(\theta^G)_N$ is a multiple of θ. In fact, a comparison of degrees yields $(\theta^G)_N = |G : N|\theta$.

Proof: Since θ is the restriction to N of a character of G, we see that θ is invariant in G, and it follows that $(\theta^G)_N$ is a multiple of θ. In fact, a comparison of degrees yields $(\theta^G)_N = |G:N|\theta$.

We have

$$[\theta^G, \theta^G] = [\theta, (\theta^G)_N] = |G:N|[\theta, \theta] = |G:N|\,.$$

Proof: Since θ is the restriction to N of a character of G, we see that θ is invariant in G, and it follows that $(\theta^G)_N$ is a multiple of θ. In fact, a comparison of degrees yields $(\theta^G)_N = |G:N|\theta$.

We have

$$[\theta^G, \theta^G] = [\theta, (\theta^G)_N] = |G:N|[\theta, \theta] = |G:N|\,.$$

Also

$$\theta^G = (1_N\theta)^G = (1_N\chi_N)^G = (1_N)^G\chi\,.$$

The irreducible constituents of $(1_N)^G$ are the members of $\mathrm{Irr}(G|1_N)$. These are exactly the characters $\beta \in \mathrm{Irr}(G)$ with $N \subseteq \ker(\beta)$.

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In other words, the irreducible constituents of $(1_N)^G$ are exactly the members of $\mathrm{Irr}(G/N)$.

If $\beta \in \mathrm{Irr}(G/N)$, then

$$[(1_N)^G, \beta] = [1_N, \beta_N] = [1_N, \beta(1)1_N] = \beta(1)\,,$$

and thus

$$(1_N)^G = \sum_{\beta \in \mathrm{Irr}(G/N)} \beta(1)\beta\,.$$

We now have

$$\theta^G = (1_N)^G \chi = \sum_{\beta \in \mathrm{Irr}(G/N)} \beta(1)\beta\chi\,,$$

so

$$|G:N| = [\theta^G, \theta^G] = \sum_{\beta,\gamma} \beta(1)\gamma(1)[\beta\chi, \gamma\chi]\,.$$

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Now $[\beta\chi, \gamma\chi] \geq 0$, and $[\beta\chi, \beta\chi] \geq 1$, and thus

$$|G : N| \geq \sum_{\beta \in \mathrm{Irr}(G/N)} \beta(1)^2\,,$$

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with equality iff $[\beta\chi, \gamma\chi] = 0$ when $\beta \neq \gamma$ and $[\beta\chi, \beta\chi] = 1$.

In fact, $\sum \beta(1)^2 = |G/N| = |G : N|$, and we deduce that the characters $\beta\chi$ are distinct and irreducible.

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The map $\beta \mapsto \beta\chi$ is thus an injection from $\mathrm{Irr}(G/N)$ into $\mathrm{Irr}(G|\theta)$.

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The map $\beta \mapsto \beta\chi$ is thus an injection from $\mathrm{Irr}(G/N)$ into $\mathrm{Irr}(G|\theta)$.

For surjectivity, recall that

$$\theta^G = \sum \beta(1)\beta\chi\,,$$

and thus the irreducible constituents of θ^G are exactly the characters $\beta\chi$. ∎

EXERCISE (7.4): Write $b(G)$ for the largest member of the set $\mathrm{cd}(G)$ of irreducible character degrees of G.

(a) If $H \subseteq G$, show that $b(H) \leq b(G)$.

(b) If $N \triangleleft G$ and G/N is nonabelian, show that $b(N) \leq b(G)/2$.

EXERCISE (7.5): Let G be solvable. Show that the derived length of G is at most $1 + 2\log(b)$, where $b = b(G)$ (as in the previous problem) and the logarithm is base 2.

THEOREM (Clifford correspondence): *Given* $\theta \in \mathrm{Irr}(N)$, *where* $N \triangleleft G$, *let* T *be the stabilizer of* θ *in* G. *Then the map* $\eta \mapsto \eta^G$ *is a bijection from* $\mathrm{Irr}(T|\theta)$ *onto* $\mathrm{Irr}(G|\theta)$. *Also, if* $\eta \in \mathrm{Irr}(T|\theta)$ *and* $\eta^G = \chi$, *then* $[\eta_N, \theta] = [\chi_N, \theta]$.

THEOREM (Clifford correspondence): *Given* $\theta \in \mathrm{Irr}(N)$, *where* $N \lhd G$, *let* T *be the stabilizer of* θ *in* G. *Then the map* $\eta \mapsto \eta^G$ *is a bijection from* $\mathrm{Irr}(T|\theta)$ *onto* $\mathrm{Irr}(G|\theta)$. *Also, if* $\eta \in \mathrm{Irr}(T|\theta)$ *and* $\eta^G = \chi$, *then* $[\eta_N, \theta] = [\chi_N, \theta]$.

Proof: Let $\eta \in \mathrm{Irr}(T|\theta)$. By Clifford's theorem, $\eta_N = e\theta$ for some positive integer e.

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Now let $\chi \in \mathrm{Irr}(G)$ lie over η, so $\chi \in \mathrm{Irr}(G|\theta)$.

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Now let $\chi \in \mathrm{Irr}(G)$ lie over η, so $\chi \in \mathrm{Irr}(G|\theta)$.

By Clifford's theorem, there is a positive integer f such that

$$\chi_N = f \sum_{i=1}^{t} \theta_i \,,$$

where the $\theta_i \in \mathrm{Irr}(N)$ are distinct and form the full G-orbit of θ. Note that $t = |G : T|$.

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where the $\theta_i \in \mathrm{Irr}(N)$ are distinct and form the full G-orbit of θ. Note that $t = |G : T|$.

Since η is a constituent of χ_T, we have $e \leq f$. Also, χ is a constituent of η^G, so $\chi(1) \leq \eta^G(1)$.

We conclude that

$$
\begin{aligned}
ft\theta(1) = \chi(1) \le \eta^G(1) &= |G:T|\eta(1) \\
&= t\eta(1) \\
&= et\theta(1) \\
&\le ft\theta(1)\,.
\end{aligned}
$$

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Equality thus holds throughout, and we deduce that $\eta^G = \chi$ and $e = f$.

$$
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\end{aligned}
$$

Equality thus holds throughout, and we deduce that $\eta^G = \chi$ and $e = f$.

Thus $\eta \mapsto \eta^G$ carries $\mathrm{Irr}(T|\theta)$ to $\mathrm{Irr}(G|\theta)$, and if $\eta^G = \chi$, then $[\eta_N, \theta] = e = f = [\chi_N, \theta]$, as required.

It remains to show that the map $\eta \mapsto \eta^G$ is a bijection from $\mathrm{Irr}(T|\theta)$ onto $\mathrm{Irr}(G|\theta)$.

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For injectivity, suppose that $\eta, \psi \in \mathrm{Irr}(T|\theta)$ and $\eta^G = \chi = \psi^G$.

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For injectivity, suppose that $\eta, \psi \in \mathrm{Irr}(T|\theta)$ and $\eta^G = \chi = \psi^G$.

Now η and ψ are constituents of χ_T, so if $\eta \neq \psi$, then

$$[\chi_N, \theta] \geq [\eta_N, \theta] + [\psi_N, \theta] = [\chi_N, \theta] + [\chi_N, \theta],$$

and this is a contradiction since $[\chi_N, \theta] > 0$.

Finally, for surjectivity, let $\chi \in \mathrm{Irr}(G|\theta)$.

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Since χ lies over θ, it lies over some character $\eta \in \mathrm{Irr}(T)$ such that η lies over θ.

Then χ is a constituent of η^G, and since we know that η^G is irreducible, we have $\eta^G = \chi$, so our map carries η to χ. $\blacksquare$

EXERCISE (7.6): Let $\theta \in \mathrm{Irr}(N)$, where $N \triangleleft G$. Show that θ^G is irreducible iff N is the full stabilizer of θ in G.

EXERCISE (7.7): A character $\chi \in \mathrm{Irr}(G)$ is primitive if there does not exist $H < G$ and a character ψ of H such that $\chi = \psi^G$. Suppose that $\chi \in \mathrm{Irr}(G)$ is primitive.

(a) If $N \lhd G$, show that χ_N is a multiple of an irreducible character of N.

(b) Let $A \lhd G$ with A is abelian. Show that $A \subseteq \mathbf{Z}(\chi)$.

Note: It follows that if G has a faithful primitive character, then $\mathbf{Z}(G)$ is cyclic and is the unique largest abelian normal subgroup of G.

EXERCISE (7.8): Show that a primitive character of a p-group must be linear.

HINT: If $\chi \in \mathrm{Irr}(G)$ then when χ is viewed as a character of $G/\ker(\chi)$, it is faithful. Also, if χ is primitive as a character of G, it is also primitive as a character of $G/\ker(\chi)$.

Chapter 8:

THEOREMS OF ITÔ AND MICHLER

THEOREM (Itô): *Assume that G is solvable. Then G has an abelian normal Sylow p-subgroup iff no member of $\mathrm{cd}(G)$ is divisible by p.*

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In this chapter, we prove Itô's theorem, and we discuss its generalization (proved by Michler) to the case where G is not necessarily solvable.

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In this chapter, we prove Itô's theorem, and we discuss its generalization (proved by Michler) to the case where G is not necessarily solvable.

Michler's argument relies on the classification of simple groups and a case-by-case study of the various simple groups. This type of argument has (unfortunately) become a standard method for proving results in character theory.

First, we discuss the assertion that if G has a normal abelian Sylow p-subgroup, then p does not divide any member of $\mathrm{cd}(G)$.

More generally, we have the following:

THEOREM (Itô divisibility): *Let $A \lhd G$ with A abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \mathrm{Irr}(G)$.*

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More generally, we have the following:

THEOREM (Itô divisibility): *Let $A \lhd G$ with A abelian. Then $\chi(1)$ divides $|G : A|$ for all $\chi \in \mathrm{Irr}(G)$.*

We have already seen that this is a consequence of the fact that $\chi(1)/\theta(1)$ divides $|G : N|$, where $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$ lies over $\theta \in \mathrm{Irr}(N)$.

This result is true in general, but we proved it only in the case where G/N is solvable.

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A different approach to Itô's divisibility theorem is to use another fact that we stated without proof: $\chi(1)$ divides $|G : \mathbf{Z}(G)|$ for all $\chi \in \mathrm{Irr}(G)$.

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Assuming this, it is easy to show that $\chi(1)$ divides $|G : \mathbf{Z}(\chi)|$ for all $\chi \in \mathrm{Irr}(G)$.

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A different approach to Itô's divisibility theorem is to use another fact that we stated without proof: $\chi(1)$ divides $|G : \mathbf{Z}(G)|$ for all $\chi \in \mathrm{Irr}(G)$.

Assuming this, it is easy to show that $\chi(1)$ divides $|G : \mathbf{Z}(\chi)|$ for all $\chi \in \mathrm{Irr}(G)$.

To see this, view χ as a character of $G/\ker(\chi)$ and observe that $\mathbf{Z}(\chi)/\ker(\chi) = \mathbf{Z}(G/\ker(\chi))$. Thus the index of the center of $G/\ker(\chi)$ is exactly $|G : \mathbf{Z}(\chi)|$.

Proof of the divisibility theorem: By hypothesis, $\chi \in \mathrm{Irr}(G)$ and $A \lhd G$ with A abelian. Let λ be a linear constituent of χ_A and let T be the stabilizer of λ in G.

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Let $\psi \in \mathrm{Irr}(T|\lambda)$ be the Clifford correspondent of χ. Then $\chi = \psi^G$, so $\chi(1) = |G : T|\psi(1)$ and it suffices to show that $\psi(1)$ divides $|T : A|$.

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Now ψ_A is a multiple of the linear character λ, so $A \subseteq \mathbf{Z}(\psi)$. We have seen that $\psi(1)$ divides $|T : \mathbf{Z}(\psi)|$, and this, in turn, divides $|T : A|$, as required ∎

We now discuss the converse. We would like to prove that if the prime p divides no member of $\mathrm{cd}(G)$, then a Sylow p-subgroup of G is normal and abelian.

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If G is simple and has no irreducible character of degree divisible by p, we would want it to be true that G has a normal Sylow p-subgroup. If G has prime order, this clearly is true, but if G is nonabelian, the only way this could happen is if p does not divide $|G|$, so the identity is a normal Sylow p-subgroup.

Let us say that a nonabelian simple group G is bad for the prime p if p divides $|G|$ but p divides no member of $\mathrm{cd}(G)$.

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If the fully general version of Itô's theorem is true, therefore, it would have to be the case that bad simple groups do not exist.

Perhaps surprisingly, the converse of this statement is also true. If there do not exist any bad simple groups, then the fully general Itô theorem holds.

In fact more is true.

THEOREM: *Let p be a prime that divides no member of* $\mathrm{cd}(G)$*, and suppose that no composition factor of G is bad for p. Then G has an abelian normal Sylow p-subgroup.*

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The original Itô theorem (for solvable groups) is an immediate consequence.

Using the classification, Michler showed that bad simple groups do not exist, so solvability in Itô's theorem is unnecessary. This proves what is called the Itô-Michler theorem.

We will need the following.

LEMMA: *Let P be a p-group acting via automorphisms on a group N, where p does not divide $|N|$, and assume that P fixes all members of* $\mathrm{Irr}(N)$. *Then the action of P on N is trivial.*

We will need the following.

LEMMA: *Let P be a p-group acting via automorphisms on a group N, where p does not divide $|N|$, and assume that P fixes all members of* $\mathrm{Irr}(N)$*. Then the action of P on N is trivial.*

Proof: In general, an automorphism of N fixes equal numbers of irreducible characters of N and classes of N. (This follows by Brauer's character table permutation lemma.)

If $x \in P$, then by hypothesis, x fixes all irreducible characters of N, and thus it fixes all classes of N.

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Let K be a class of N. Then P acts on K, and since $|K| \not\equiv 0 \bmod p$, it follows that P fixes an element of K. Thus $K \cap C \neq \emptyset$, where $C = \mathbf{C}_N(P)$.

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It follows that $K \subseteq \bigcup C^x$, where x runs over G. Then $N = \bigcup C^x$, and we argue that $C = N$.

Otherwise, there must be more than one conjugate of C in N so

$$|N| = \Big|\bigcup_{x \in N} C^x\Big| < \sum_{x \in N} |C^x| \leq |N : C||C| = |N|,$$

and this is a contradiction. Thus $C = N$, so the action of P is trivial. ∎

Proof of composition factor theorem: If $N \triangleleft G$ and $\theta \in \mathrm{Irr}(N)$, let $\chi \in \mathrm{Irr}(G)$ lie over θ. Then $\theta(1)$ divides $\chi(1)$ by Clifford's theorem, so p does not divide $\theta(1)$, and thus N satisfies the hypotheses of the theorem. Also G/N satisfies the hypothesis since irreducible characters of G/N can be viewed as irreducible characters of G.

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If $P \triangleleft G$, where P is a p-group then $\mathrm{cd}(P)$ consists of powers of p. It follows by the above that $\mathrm{cd}(P) = \{1\}$, so P is abelian. It suffices, therefore, to show that G has a normal Sylow p-subgroup.

There is nothing to prove if G is trivial, so we can assume $G > 1$, and we induct on $|G|$. Let N be maximal normal in G.

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If $P > 1$, the inductive hypothesis says that G/P has a normal Sylow p-subgroup S/P. Then S is a normal Sylow p-subgroup of G, as wanted

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We can now assume that $P = 1$, and thus p does not divide $|N|$.

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Now G/N is a simple group, and no member of $\mathrm{cd}(G/N)$ is divisible by p. Also, p divides $|G/N|$, and by hypothesis, G/N is not bad. It follows that G/N is not a nonabelian simple group, and we conclude that $|G/N| = p$.

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Let $\theta \in \mathrm{Irr}(N)$. Then θ must be G-invariant since otherwise, N is the stabilizer of θ, and thus θ^G is irreducible with degree divisible by p.

Let S be a Sylow p-subgroup of G. Then $G = NS$ and S fixes all members of $\mathrm{Irr}(N)$, so by the lemma, S acts trivially on N. Then

$$N \subseteq \mathbf{C}_G(S) \subseteq \mathbf{N}_G(S)\,,$$

and hence $S \triangleleft G$. ∎

Chapter 9:

M-GROUPS

A character χ of a group G is monomial if there exists a subgroup $H \subseteq G$ and a linear character λ of H such that $\lambda^G = \chi$.

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Note: Every linear character of G is monomial. (Simply take H to be G in the above definition.)

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Note: Every linear character of G is monomial. (Simply take H to be G in the above definition.)

A group G is an M-group if every irreducible character of G is monomial.

THEOREM (Taketa): *Let G be an M-group. Then G is solvable and the derived length of G is at most $|\mathrm{cd}(G)|$.*

Digression on solvable groups:

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The derived subgroup (also called the commutator subgroup) of G is the subgroup generated by all elements of G of the form $[x, y] = x^{-1}y^{-1}xy$.

This characteristic subgroup is usually denoted G', but sometimes the notation $G^{(1)}$ is used.

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Note: G is abelian iff $G' = 1$.

The derived subgroup G'' of G' is sometimes denoted $G^{(2)}$ and its derived subgroup G''' is usually denoted $G^{(3)}$, and so on.

The derived series of G is the chain

$$G = G^0 \supseteq G^{(1)} \subseteq G^{(2)} \supseteq G^{(3)} \supseteq \cdots .$$

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Note that all terms of the derived series of G are characteristic in G.

If $G^{(m)} = 1$ for some integer m, then G is solvable and the derived length of G (denoted $\mathrm{dl}(G)$) is the smallest integer m such that $G^{(m)} = 1$.

There are several properties equivalent (for finite groups) to solvability. Sometimes one or another of these is used as the definition.

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Exercise (9.1): Let $N \lhd G$. Show that G is solvable iff both N and G/N are solvable, and in this case, show that $\mathrm{dl}(G) \leq \mathrm{dl}(N) + \mathrm{dl}(G/N)$.

Exercise (9.2): Let G be finite. Show that the following are equivalent.

(1) G is solvable.
(2) The chief factors of G are abelian.
(3) The composition factors of G are cyclic.

We need the following for Taketa's theorem.

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Proof: If $g \in \ker(\alpha^G)$, then $\alpha^G(g) = \alpha^G(1)$ so

$$\sum_{x \in G} \alpha^0(xgx^{-1}) = \sum_{x \in G} \alpha^0(1) = \sum_{x \in G} \alpha(1)\,.$$

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But $|\alpha^0(xgx^{-1})| \leq \alpha(1)$ for all $x \in G$, so equality holds for all x.

Then $|\alpha^0(g)| = \alpha(1) \neq 0$, so $g \in H$. ∎

Proof of Taketa's theorem: Write

$$\mathrm{cd}(G) = \{f_1, f_2, f_3, \ldots, f_r\},$$

where

$$1 = f_1 < f_2 < f_3 \cdots < f_r .$$

Note that $r = |\mathrm{cd}(G)|$.

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We will show by induction on k that if $\chi \in \mathrm{Irr}(G)$ and $\chi(1) = f_k$, then $G^{(k)} \subseteq \ker(\chi)$.

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Note that $r = |\mathrm{cd}(G)|$.

We will show by induction on k that if $\chi \in \mathrm{Irr}(G)$ and $\chi(1) = f_k$, then $G^{(k)} \subseteq \ker(\chi)$.

It will then follow that $G^{(r)} \subseteq \ker(\chi)$ for all $\chi \in \mathrm{Irr}(G)$, so $G^{(r)} = 1$. Then G is solvable and $\mathrm{dl}(G) \leq r = |\mathrm{cd}(G)$, as required.

We are given $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = f_k$. If $k = 1$, then χ is linear, so $G' \subseteq \ker(\chi)$, and this is the base case of the induction.

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Now assume $k > 1$. Since χ is monomial, we can write $\chi = \lambda^G$, where λ is a linear character of G $H \subseteq G$.

We are given $\chi \in \operatorname{Irr}(G)$ with $\chi(1) = f_k$. If $k = 1$, then χ is linear, so $G' \subseteq \ker(\chi)$, and this is the base case of the induction.

Now assume $k > 1$. Since χ is monomial, we can write $\chi = \lambda^G$, where λ is a linear character of some subgroup $H \subseteq G$.

We have

$$f_k = \chi(1) = \lambda^G(1) = |G : H|\lambda(1) = |G : H|\,.$$

Now

$$(1_H)^G(1) = |G : H| = f_k \,,$$

so $(1_H)^G = 1_G + \Xi$, where each irreducible constituent of Ξ has degree smaller that f_k.

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By the inductive hypothesis, $G^{(k-1)}$ is contained in the kernel of every irreducible constituent of $(1_H)^G$.

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so $(1_H)^G = 1_G + \Xi$, where each irreducible constituent of Ξ has degree smaller that f_k.

By the inductive hypothesis, $G^{(k-1)}$ is contained in the kernel of every irreducible constituent of $(1_H)^G$. Thus

$$G^{(k-1)} \subseteq \ker((1_H)^G) \subseteq H\,,$$

where the final containment is by the lemma.

We thus have

$$G^{(k)} = (G^{(k-1)})' \subseteq H' \subseteq \ker(\lambda) \,.$$

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Now χ lies over λ and λ lies over the principal character of $G^{(k)}$, so χ lies over $1_{G^{(k)}}$.

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Now χ lies over λ and λ lies over the principal character of $G^{(k)}$, so χ lies over $1_{G^{(k)}}$.

By Clifford's theorem, therefore, the principal character of $G^{(k)}$ is the unique irreducible constituent of $\chi_{G^{(k)}}$.

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Then $G^{(k)} \subseteq \ker(\chi)$, as required. $\blacksquare$

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By Clifford's theorem, therefore, the principal character of $G^{(k)}$ is the unique irreducible constituent of $\chi_{G^{(k)}}$.

Then $G^{(k)} \subseteq \ker(\chi)$, as required. ∎

Open question: Is it true that $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ for general solvable groups?

EXERCISE (9.3): Show that nilpotent groups are M-groups.

HINT: Show that a primitive irreducible character of a nilpotent group G must be linear. To do this, assume $\chi \in \mathrm{Irr}(G)$ is not linear and consider a subgroup $N \triangleleft G$ maximal with the property that χ_N is reducible. Show that χ_N is not a multiple of an irreducible character.

EXERCISE (9.4): Suppose G' is abelian. Show that G is an M-group.

EXERCISE (9.5): Let $\chi \in \mathrm{Irr}(G)$ be monomial, and suppose that $A \triangleleft G$ with A abelian. Show that there exists a subgroup H with $A \subseteq H \subseteq G$ and a linear character λ of H such that $\lambda^G = \chi$.

HINTS: We have $\chi = \mu^G$ where μ is a linear character of some subgroup $J \subseteq G$. Let $K = AJ$ and $\psi = \mu^K$ and note that ψ is irreducible. Let $\theta \in \mathrm{Irr}(A)$ lie under ψ. Use Exercise 6.1 to show that $[\psi_A, \theta] = 1$. Use the Clifford correspondence in the group K.

Every M-group is solvable, but not every solvable group is an M-group. The smallest example of a solvable group that is not an M-group is $SL(2,3)$, which has order 24. We have seen that every nilpotent group is an M-group, so the class of M-groups fits somehow between "nilpotent" and "solvable".

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In fact, there is a class of groups called "supersolvable" groups, lying between nilpotent and solvable. All supersolvable groups are M-groups, but not conversely.

A theorem of Dade shows that every solvable group is a subgroup of an M-group. It follows that M-groups can have non-M-subgroups, but certain subgroups of M-groups can be proved to be M-groups.

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THEOREM: *Let G be an M-group and suppose that $N \triangleleft G$ and that $|N|$ and $|G : N|$ are relatively prime. Then N is an M-group.*

A theorem of Dade shows that every solvable group is a subgroup of an M-group. It follows that M-groups can have non-M-subgroups, but certain subgroups of M-groups can be proved to be M-groups.

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This suggests two interesting questions:

Can we drop the normality hypothesis?
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The Dade/van der Waall example involves the prime 2 in an essential way. It is unknown if a normal subgroup of an M-group having odd order or odd index must be an M-group.

Proof of theorem: Let $\theta \in \mathrm{Irr}(N)$, so we must show that θ is monomial. Let $\chi \in \mathrm{Irr}(G)$ lie over θ and write $\chi = \lambda^G$, where λ is a linear character of some subgroup H of G.

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Now $(\lambda^{NH})^G = \lambda^G = \chi$ and this is irreducible. Thus λ^{NH} is irreducible. We have $\lambda^{NH}(1) = |NH : H| = |N : N \cap H|$, and this divides $|N|$.

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Then λ^{NH} is an irreducible character of NH with degree relatively prime to $|NH : N|$ and thus $(\lambda^{NH})_N$ is irreducible.

Write

$$\varphi = (\lambda^{NH})_N = (\lambda_{N\cap H})^N ,$$

so φ is monomial.

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Now φ lies under λ^{NH} and λ^{NH} lies under χ. By Clifford's theorem, it follows that that φ and θ are conjugate in G. Since φ is monomial, θ is also monomial. $\blacksquare$