MODULE OF DIFFERENTIALS, EVOLUTIONS, REDUCTIONS, AND BRIANÇON SKODA THEOREM

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CONTENTS

1. DERIVATIONS AND THE MODULE OF DIFFERENTIALS

Let R be a ring and M an R-module.

A map $d : R \longrightarrow M$ is a *derivation* if for every x, y in R

(1) $d(x+y) = d(x) + d(y)$ (d is a group homomorphism)
(2) $d(xy) = xd(y) + yd(x)$ (product rule)

Examples 1.1. (1) The trivial map $d : R \longrightarrow M$ such that $d(r) = 0$ for all $r \in R$ is a derivation, called *the trivial derivation*. There are examples of rings and modules where this is the only derivation, for instance $\mathbb{Z}[i]$.

(2) Let $R = k[x_1, \dots, x_n]$ be a polynomial ring. Then the usual partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$ are derivations.

Remark 1.2. $\ker d$ is a subring of R. $[1 \in \ker d$ as $d(1) = d(1 \cdot 1) = d(1) + d(1)]$

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Let $\varphi : A \longrightarrow R$ be a morphism so that R is an A-algebra. A derivation $d : R \longrightarrow M$ is a *derivation over A* if in addition

(3) $\varphi(A) \subset \ker d$.

Remark 1.3. Let $\varphi : A \longrightarrow R$ be a morphism so that R is an A-algebra.

(a) $\varphi(A) \subset \ker d \Longleftrightarrow d$ is A-linear.
(b) Every derivation is a derivation over $\mathbb{Z}$.
(c) Two derivations over A coincide if they have the same values on a generating set of R as an A-algebra.

Proof. (a): $\Longrightarrow d(ax) = d(\varphi(a)x) = \varphi(a)d(x) + xd(\varphi(a)) = \varphi(a)d(x) = ad(x)$.
$\Longleftarrow$ For all a in A, $\quad d(\varphi(a)) = d(a \cdot 1) = ad(1) = 0$.
(b), (c): Clear since the kernel of any derivation is a subring. □

The set of all A-derivations forms an R-module, denoted

$$\mathrm{Der}_A(R,M) = \{d : R \to M \,|\, d \text{ a derivation over } A\}$$

The R-module $\mathrm{Der}_A(R) = \mathrm{Der}_A(R,R)$ is called the *module of derivations of R over A*.

Example 1.4. $R = A[x_1,\ldots,x_n]$ a polynomial ring. Then $\partial_i = \frac{\partial}{\partial x_i} \in \mathrm{Der}_A(R)$ and $\{\partial_1,\ldots,\partial_n\}$ form a basis of $\mathrm{Der}_A(R)$ as an R-module.

Proof. To show this is a generating set let $d \in \mathrm{Der}_A(R)$. Then $d = d(x_1)\partial_1 + \ldots + d(x_n)\partial_n$ are equal by 1.3(c) since both derivations have the same values on $x_1,\ldots,x_n$. To show linear independence let $\lambda_1\partial_1 + \ldots + \lambda_n\partial_n = 0$ with $\lambda_i \in R$. Evaluating at x_i we get $\lambda_i = 0$. □

Remark 1.5. (a) If $d \in \mathrm{Der}(R,M) = \mathrm{Der}_{\mathbb{Z}}(R,M)$ then $d \cdot \varphi \in \mathrm{Der}(A,M)$.
(b) If $d \in \mathrm{Der}_A(R,M)$ and $\Psi : M \longrightarrow N$ is a homomorphism of R-modules then $\Psi \cdot d \in \mathrm{Der}_A(R,N)$.

The module of derivations is very difficult to compute. Instead we focus our attention on the *module of differentials of R over A*, denoted $\Omega_A(R)$. We will see later that its dual is the module of derivations. The latter is easy to compute given a presentation of R. Furthermore it is defined by a universal property that shows its independence from the presentation of R. We will give three different definitions of the modules of differentials. First via the universal property, second, concretely, in terms of generators and relations, and third through the diagonal ideal.

Theorem 1.6 (Universal property of the module of differentials)**.** *Let R be an A-algebra*

(a) There exist an R-module $\Omega_A(R)$ and a derivation $d_{R/A} : R \longrightarrow \Omega_A(R)$ over A having this property: For every derivation $\delta : R \longrightarrow M$ over A there exists a unique R-linear map

$$\Phi : \Omega_A(R) \longrightarrow M \quad \textit{with} \quad \delta = \Phi \cdot d_{R/A}$$

$$\begin{array}{ccc} R & \xrightarrow{\;d_{R/A}\;} & \Omega_A(R) \\ & {\scriptstyle\delta}\searrow \qquad \swarrow{\scriptstyle\exists!\Phi} & \\ & M & \end{array}$$

(*b*) *The universal property of* (*a*) *determines* $(\Omega_A(R), d_{R/A})$ *uniquely up to canonical R-isomorphism. That is if* (Ω', d') *has the same property, then there is a unique R-isomorphism*

$$\Psi : \Omega_A(R) \xrightarrow{\cong} \Omega' \quad \textit{with} \quad d' = \Psi \cdot d_{r/A}\,.$$

Proof. (b) is clear by the standard diagram chase for universal objects.

(a) **Case 1**: If $R = A[\{x_i \,|\, i \in \mathfrak{I}\}]$ is a polynomial ring over A, then we take $\Omega_A(R)$ to be the free R-module with basis $\{dx_i \,|\, i \in \mathfrak{I}\}$ and we take the derivation $d_{R/A} : R \longrightarrow \Omega_A(R)$ to be the map

$$d_{R/A}(f) = \sum_i \frac{\partial f}{\partial x_i} dx_i\,.$$

Notice that this sum is finite as f involves only finitely many variables, hence $d_{R/A}(f) \in \Omega_A(R)$. As seen in Example 1.4, $d_{R/A}$ is a well defined derivation over A, with $d_{R/A}(x_i) = dx_i$. We must show that $(\Omega, d_{R/A})$ has the universal property described above.

Let $\delta : R \longrightarrow M$ be any derivation over A. Since $\Omega_A(R)$ is a free R-module with basis $\{dx_i \,|\, i \in \mathfrak{I}\}$ there exists an R-linear map $\Phi : \Omega_A(R) \longrightarrow M$ with $\Phi(dx_i) = \delta(x_i)$. Hence $\delta(x_i) = \Phi \cdot d_{R/A}(x_i)$ for all $i \in \mathfrak{I}$. But δ and $\Phi \cdot d_{R/A}$ are both derivations over A by Remark 1.5(b), and $\{x_i \,|\, i \in \mathfrak{I}\}$ generates R as an A-algebra. Hence, according to Remark 1.3(c), $\delta = \Phi \cdot d_{R/A}$. Furthermore, Φ is uniquely determined because $Rd_{R/A}(R) = \Omega_A(R)$, in other words Ω is generated by the image of R.

Case 2: [general case] Let $R = S/I$ with $S = A[\{x_i \,|\, i \in \mathfrak{I}\}]$ a polynomial ring over A. Write $D = d_{S/A} : S \longrightarrow \Omega_A(S)$ and take $(\Omega, d_{R/A})$ as

$$\Omega_A(R) = \Omega_A(S)/SD(I) + I\Omega_A(S) = \Omega_A(S)/SD(I) + ID(S)\,.$$

To define the derivation $d_{R/A}$ we consider the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad D \quad} & \Omega_A(S) \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle p} \\ R = S/I & \overset{\overline{D}}{\dashrightarrow} & \Omega_A(S)/SD(I) + ID(S) \end{array}$$

where the induced map $\overline{D}$ exists as a homomorphism of groups since $I \subset \ker(p \cdot D)$. Since D is a derivation over A, $\overline{D}$ is. We set

$$d_{R/A} = \overline{D}$$

Now we show that the pair $(\Omega, d_{R/A})$ has the universal property. Let $\delta : R \longrightarrow M$ be any derivation over A. Then $\delta \cdot \pi : S \longrightarrow M$ is a derivation over A by Remark 1.5(a). Thus by Case 1, there exists an S-linear map $\Phi : \Omega_A(S) \longrightarrow M$ with $\delta \cdot \pi = \Phi \cdot D$. Now $\Phi(D(I)) = \delta(\pi(I)) = 0$, hence $\Phi(SD(I)) = 0$. Also, $\Phi(ID(S)) = I\Phi(D(S)) = 0$ since M is an R-module. So Φ induces the R-linear map $\overline{\Phi}:\Omega_A(S)/SD(I)+ID(S)\longrightarrow M$ with $\Phi=\overline{\Phi}\cdot p$. Furthermore, $\delta\cdot\pi=\Phi\cdot D=\overline{\Phi}\cdot p\cdot D=\overline{\Phi}\cdot\overline{D}\cdot\pi$. Hence $\delta=\overline{\Phi}\cdot\overline{D}=\overline{\Phi}\cdot d_{R/A}$. Finally $\overline{\Phi}$ is uniquely determined because $Rd_{R/A}(R)=\Omega_A(R)$. □

Definition 1.7 (The module of differentials)**.** The R-module $\Omega_A(R)$, whose existence has been shown in Theorem 1.6, is called the *module of differentials of R over A*, or, *universal module of differentials*, or, *module of Kähler differentials*. The derivation $d_{R/A}:R\longrightarrow\Omega_A(R)$ is called the *universal derivation of R over A*.

Remark 1.8. (a) $\mathrm{Der}_A(R,M)\underset{\mathrm{nat}}{\cong}\mathrm{Hom}_R(\Omega_A(R),M)$ via Theorem 1.6(a).

(b) If $R=S/I$ with $S=A[\{x_i\,|\,i\in\mathfrak{I}\}]$ a polynomial ring over A and $I=(f_j\,|\,j\in\mathcal{J})$ and S-ideal, then

$$\Omega_A(R)\cong\bigoplus_{i\in\mathfrak{I}}Rdx_i/(\sum\overline{\frac{\partial f}{\partial x_i}}dx_i\,|\,f\in I)$$
$$=\bigoplus_{i\in\mathfrak{I}}Rdx_i/(\sum\overline{\frac{\partial f_j}{\partial x_i}}dx_i\,|\,j\in\mathcal{J})$$

where $\{dx_i\,|\,i\in\mathfrak{I}\}$ is an R-basis and $^-$ denotes images in R. Moreover $d_{R/A}(\overline{x_i})$ is the image of dx_i in $\Omega_A(R)$.

(c) The R-module $\Omega_A(R)$ is generated as an R-module by the image $d_{R/A}(R)$,

$$\Omega_A(R)=Rd_{R/A}(R)\,.$$

Proof. (b) The first isomorphism and the statement about $d_{R/A}$ follows from the proof of Theorem 1.6. The second equality holds because for every $s\in S$ we have

$$\overline{\frac{\partial(sf_j)}{\partial x_i}}=\overline{s}\overline{\frac{\partial f_j}{\partial x_i}}+\overline{\frac{\partial s}{\partial x_i}}\overline{f_j}=\overline{s}\overline{\frac{\partial f_j}{\partial x_i}}$$

since the f_j's are in I. □

Notice: $\Omega_A(R)$ is finitely generated as an R-module if R is finitely generated as an A-algebra.

Example 1.9. If $R=A[x]/(f)$ is a hypersurface then $\Omega_A(R)=Rdx/Rf'dx\cong R/(f')$, where f' is the derivative of f with respect to x.

Theorem 1.10 (Third construction of the module of differentials)**.** *Let R be an A-algebra and consider the exact sequence*

$$0\longrightarrow\mathbb{D}\longrightarrow R\otimes_A R\stackrel{\mu}{\longrightarrow}R\longrightarrow 0$$

where μ is the multiplication map and $R\otimes_A R$ is the enveloping algebra. Then the R-linear map

$$\delta:R\longrightarrow\mathbb{D}/\mathbb{D}^2\qquad\text{with}\quad\delta(x)=\overline{(x\otimes 1)-(1\otimes x)}\in\mathbb{D}/\mathbb{D}^2$$

is the universal derivation of R over A. Thus $\mathbb{D}/\mathbb{D}^2$ is isomorphic to the module of differentials of R over A.

Proof. We will show that the pair $(\mathbb{D}/\mathbb{D}^2, \delta)$ is in a natural sense isomorphic to the pair $(\Omega_A(R), d_{R/A} = d)$. Clearly $\mathbb{D}/\mathbb{D}^2$ is an R-module (quite generally, if $S \longrightarrow R$ is an algebra map whose kernel J has square 0 then J is naturally an R-module) and δ is a derivation over A. Indeed

$$\begin{aligned}\delta(xy) &= \overline{(xy\otimes 1)-(1\otimes xy)} = \overline{(xy\otimes 1)-x\otimes y+x\otimes y-(1\otimes xy)} \\ &= x\overline{(y\otimes 1)-(1\otimes y)}+y\overline{(x\otimes 1)-(1\otimes x)} = x\delta(y)+y\delta(x)\end{aligned} \tag{1}$$

and δ is obviously A-linear.

Thus by the universal property of $(\Omega_A(R), d)$, there exists a unique R-homomorphism $\varphi : \Omega_A(R) \longrightarrow \mathbb{D}/\mathbb{D}^2$ satisfying $\delta = \varphi \cdot d$; that is, with

$$\varphi(dx) = \overline{(x\otimes 1)-(1\otimes x)}.$$

It is enough to show that φ is an isomorphism. We construct an inverse of φ. Let S be the trivial extension of R by $\Omega_A(R)$, that is, the ring that as an abelian group is the direct sum of R and $\Omega_A(R)$ and whose multiplication is defined for all $x, y \in R$ and $a, b \in \Omega_R(A)$ to be $(x,a)(y,b) = (xy, xb+ya)$. We first consider the two maps of A-algebras $\Psi_1 : R \longrightarrow S$ and $\Psi_2 : R \longrightarrow S$ given by $\Psi_1(x) = (x, dx)$ and $\Psi_2(y) = (y, 0)$, respectively. By the universal property of the tensor product we then obtain the map of A-algebras $\Psi : R \otimes_A R \longrightarrow S$ given by $\Psi(x \otimes y) = \Psi_1(x)\Psi_2(y) = (xy, ydx)$. The restriction of Ψ to $\mathbb{D}$ induces the desired inverse of φ since $\Psi(x \otimes 1 - 1 \otimes x) = (x, dx) - (x, xd1) = (0, dx)$. □

Proposition 1.11 (Localization)**.** *Let R be an A-algebra, W a multiplicatively closed subset of R, $d = d_{R/A}$. Then*

$$\Omega_A(W^{-1}R) = W^{-1}\Omega_A(R)$$

and for $r \in R$, $w \in W$ we have

$$d_{W^{-1}R/A}(\frac{r}{w}) = \frac{wdr - rdw}{w^2}$$

Proof. Check that $d_{W^{-1}R/A} : W^{-1}R \longrightarrow W^{-1}\Omega_A(R)$ is well defined, is a derivation over A, and the pair $(W^{-1}\Omega_A(R), d_{W^{-1}R/A})$ has the universal property of Theorem 1.6. □

Corollary 1.12. *If $R = W^{-1}(S/I)$ with $S = A[\{x_i \mid i \in \mathfrak{I}\}]$ a polynomial ring, $W \subset S$ a multiplicative set, and $I = (f_j \mid j \in \mathcal{J})$ an S-ideal. Then*

$$\Omega_A(R) \cong \bigoplus_{i\in\mathfrak{I}} R dx_i / (\sum \overline{\frac{\partial f_j}{\partial x_i}} dx_i \mid j \in \mathcal{J})$$

and $d_{R/A}(\overline{x_i})$ is the image of dx_i in $\Omega_A(R)$. In particular, if $\mathfrak{I}$ is finite then $\Omega_A(R)$ is a finite R-module, and if $\mathfrak{I}$ and $\mathcal{J}$ are finite, then $\Omega_A(R)$ is a finitely presented R-module, presented by the matrix

$$\left(\overline{\frac{\partial f_j}{\partial x_i}}\right)$$

The association $\begin{array}{c} R \\ {\scriptstyle\varphi}\uparrow \\ A \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \Omega_A(R) \\ \uparrow{\scriptstyle d_{R/A}} \\ R \end{array}$ is functorial:

Given a commutative diagram of homomorphism of rings

$$\begin{array}{ccc} R & \xrightarrow{\psi} & T \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

there exists a unique R-linear map Ψ by the universal property so that the diagram

$$\begin{array}{ccc} \Omega_A(R) & \xrightarrow{\Psi} & \Omega_B(T) \\ {\scriptstyle d_{R/A}}\uparrow & & \uparrow{\scriptstyle d_{T/B}} \\ R & \xrightarrow{\psi} & T \end{array}$$

commutes. Notice $\Psi(d_{R/A}(r)) = d_{T/B}(\psi(r))$. Furthermore, Ψ induces a T-linear map

$$T \otimes_R \Omega_A(R) \longrightarrow \Omega_B(T) \qquad \text{defined as} \qquad t \otimes a \to t\Psi(a)$$

Given homomorphism of rings $A \longrightarrow B \longrightarrow R$ we have two diagrams

$$\begin{array}{ccc} B & \longrightarrow & R \\ \uparrow & & \uparrow \\ A & = & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R & = & R \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

and induced R-linear maps

$$R \otimes_B \Omega_A(B) \longrightarrow \Omega_A(R) \qquad \text{and} \qquad \Omega_A(R) \longrightarrow \Omega_B(R)$$

Proposition 1.13 (Relative Cotangent Sequence)**.** *With A, B, and R as above, the sequence*

$$R \otimes_B \Omega_A(B) \longrightarrow \Omega_A(R) \longrightarrow \Omega_B(R) \longrightarrow 0$$

is exact.

Proof. Let $\varphi : B \longrightarrow R$, $\Omega = \Omega_A(R)$, and $d = d_{R/A}$. We show that the map

$$R \longrightarrow \Omega/Rd(\varphi(B)) \qquad \text{given by} \qquad r \to \overline{dr}$$

is the universal derivation of R over B. But it is a derivation over B by Remark 1.5(b) and it has the universal property of Theorem 1.6. □

Corollary 1.14. *Let $\varphi : A \longrightarrow R$ be an algebra, $V \subset A$, $W \subset R$ multiplicatively closed subsets with $\varphi(V) \subset W$. Then*

$$\Omega_{V^{-1}A}(W^{-1}R) \cong \Omega_A(W^{-1}R) \cong W^{-1}\Omega_A(R)$$

via the natural maps.

Proof. The second isomorphism holds by Proposition 1.11. The first one follows from Proposition 1.13 once we have shown that $\Omega_A(V^{-1}A) = 0$. This is clear since by Proposition 1.11, $\Omega_A(V^{-1}A) \cong V^{-1}\Omega_A(A) = 0$. □

2. The conormal sequence and the rank of the module of differentials

Consider the maps $A \longrightarrow R \longrightarrow T = R/I$. By the relative cotangent sequence 1.13 we obtain an induced map

$$T \otimes_R \Omega_A(R) \longrightarrow \Omega_A(T) \qquad \text{with} \quad t \otimes dr = tdr$$

that is surjective since $\Omega_R(T) = 0$. By the definition of the module of differentials in terms of generators and relations (see Remark 1.8), the kernel of this map is generated by the image of $d_{R/A}(I)$. This image is also the image of the map

$$\delta : I/I^2 \longrightarrow T \otimes_R \Omega_A(R) \qquad \text{given by} \quad \delta(f + I^2) = 1 \otimes df$$

This map is well-defined since $d(I^2) \subset I\Omega_A(R)$ maps to zero in $T \otimes_R \Omega_A(R)$. It is T-linear since for $r \in R$ and for $f \in I$,

$$\delta(rf + I^2) = 1 \otimes d(rf) = 1 \otimes (rdf + fdr) = 1 \otimes rdf = r\delta(f + I^2).$$

Hence we have

Proposition 2.1 (The Conormal Sequence)**.** *For $A \longrightarrow R \longrightarrow T = R/I$, the sequence*

$$I/I^2 \xrightarrow{\delta} T \otimes_R \Omega_A(R) \longrightarrow \Omega_A(T) \longrightarrow 0$$

is an exact sequence of T-linear maps. Furthemore, there is an induced exact sequence

$$I/(I \cap A)R + I^2 \xrightarrow{\bar{\delta}} T \otimes_R \Omega_A(R) \longrightarrow \Omega_A(T) \longrightarrow 0$$

It is a difficult problem to decide when the left-hand map of the conormal sequence is an injection (in a homework problem you will solve a particular case), but in general we can give a criterion that tells us when is a split injection.

Proposition 2.2. *Let $\varphi : R \longrightarrow T$ be a homomorphism of A-algebras, I a T-ideal with $I^2 = 0$, $\Delta : R \longrightarrow I$ an A-linear map. Then Δ is a derivation over A if and only if $\varphi + \Delta : R \longrightarrow T$ is a homomorphism of A-algebras.*

Proof. Let x, y be in R. Then

$$(\varphi + \Delta)(xy) = \varphi(xy) + \Delta(xy)$$

and

$$(\varphi + \Delta)(x)(\varphi + \Delta)(y) = \varphi(x)\varphi(y) + \varphi(x)\Delta(y) + \varphi(y)\Delta(x)$$

since $I^2 = 0$. □

Proposition 2.3 (Conormal sequence is split injective)**.** *The T-linear map δ in the conormal sequence has a left inverse if and only if the natural projection of A-algebras $\pi : R/I^2 \longrightarrow R/I = T$ has a right inverse.*

Proof. As $d_{R/A}(I^2) \subset I\Omega_A(R)$, by Remark 1.8 (b) we have

$$R/I \otimes_{R/I^2} \Omega_A(R/I^2) \cong R/I \otimes_{R/I^2} \Omega_A(R)/Rd_{R/A}(I^2) + I^2\Omega_A(R) \cong R/I \otimes_R \Omega_A(R)$$

Hence δ does not change when we replace R by R/I^2, and neither does π. So we may assume that $I^2 = 0$.

$\pi : R \longrightarrow R/I$ has a right inverse
$\Longleftrightarrow$ $\exists$ a homomorphism of A-algebras $\rho : R/I \longrightarrow R$ with $\pi \cdot \rho = \mathrm{id}_{R/I}$
$\Longleftrightarrow$ $\exists$ a homomorphism of A-algebras $\tau : R \longrightarrow R$ with $\tau_{|I} = 0$ and $(\mathrm{id}_R - \tau)(R) \subset I$
$\overset{2.2}{\Longleftrightarrow}$ $\exists$ a derivation $\Delta : R \longrightarrow I$ over A with $\Delta_{|I} = \mathrm{id}_I$
$\overset{1.6}{\Longleftrightarrow}$ $\exists$ an R-linear map $\Phi : \Omega_A(R) \longrightarrow I$ with $\Phi(d_{R/A}(f)) = f$ for all $f \in I$
$\Longleftrightarrow$ $\exists$ an T-linear map $\Psi : T \otimes_A \Omega_A(R) \longrightarrow I = I/I^2$ with $\Psi(1 \otimes d_{R/A}(f)) = f + I^2$ for all $f \in I$
$\Longleftrightarrow$ δ has a left inverse. □

Let $k \subset K$ be a field extension. For the rest of this section we assume that k is a field of characteristic zero.

Our goal is to compute the rank of the module of differentials. We will show that given a finitely generated field extension $k \subset K$, the module of differentials $\Omega_k(K) = 0$ if and only if the extension is algebraic. We first recall the notion of transcendence degree and transcendence basis.

A subset $U \subset K$ is a *transcendence basis* of K over k if U is algebraically independent over k and $k(U) \subset K$ is algebraic.

We recall a fact from fields theory.

Proposition 2.4. *(1) Let $V \subset W$ be (possibly empty) subsets of K so that V is algebraically independent over k and $k(W) \subset K$ is algebraic. Then there exists a transcendence basis U of K over k with $V \subset U \subset W$.*

(2) Let U, U' be transcendence bases of K over k. Then for every $u' \subset U'$ there exists $u \in U$ so that $(U' \setminus \{u'\}) \cup \{u\}$ is a transcendence basis of K over k.

(3) Any two transcendence bases of K over k have the same cardinality

It follows that any $k \subset K$ has a transcendence basis and that the cardinality of any such basis only depends on the extension $k \subset K$. This cardinality is called the *transcendence degree* of K over k, $\mathrm{trdeg}_k K$.

Corollary 2.5. *If $k \subset K \subset L$ are field extensions, then*

$$\mathrm{trdeg}_k L = \mathrm{trdeg}_k K + \mathrm{trdeg}_K L$$

Proof. If U is a transcendence basis of K over k and V is a transcendence basis of L over K, then $U \cup V$ is a transcendence basis of L over k. □

Lemma 2.6. *If $k \subset K \subset L$ are field extensions with $K \subset L$ algebraic, then*

$$L \otimes_K \Omega_k(K) \cong \Omega_k(L)$$

via the natural L-linear map.

Proof. We have $L = \cup L_i = \varinjlim L_i$, where L_i are the finite field extensions of K contained in L. Now $\varinjlim$ is compatible with $\otimes_K \Omega_K(K)$ and taking differentials (see homework 6). Hence we may replace L by L_i and assume that the extension $K \subset L$ is finite, hence simple (since in characteristic zero every algebraic extension is separable and by the primitive element theorem every separable finite extension is simple). Write $L = K[x] = K[X]/(f(X))$ with $f'(x) \neq 0$ in L. Now by Remark 1.8 (b) we have

$$\Omega_k(L) \cong L \otimes_K \Omega_k(K) \oplus Ldx/L(\star, f'(x)) \cong L \otimes_K \Omega_k(K)$$

since $f'(x)$ is a unit in L. □

Theorem 2.7. *Let $k \subset K$ be a field extension, $U = \{x_i\} \subset K$, $d = d_{K/k}$. Then U is a transcendence basis of K over k $\Longleftrightarrow$ $\{dx_i\}$ is a K-basis of $\Omega_k(K)$.*

Proof. $\Longrightarrow$ Let $K_0 = k(U)$. By Remark 1.8 (b), and Proposition 1.11, $\{d_{K_0/k}(x_i)\}$ are a K_0-basis of $\Omega_k(K_0)$. But $K_0 \subset K$ is an algebraic extension, hence $\Omega_k(K) \cong K \otimes_{K_0} \Omega_k(K_0)$ by the previous lemma. Thus $\{dx_i\} = \{1 \otimes d_{K_0/k}(x_i)\}$ form a K-basis of $\Omega_k(K)$.

$\Longleftarrow$ Let $K_0 = k(U)$. From the Relative Cotangent Sequence we have

$$K \otimes_{K_0} \Omega_k(K_0) \longrightarrow \Omega_k(K) \longrightarrow \Omega_{K_0}(K) \longrightarrow 0$$

where the first map is surjective by our assumption. Thus $\Omega_{K_0}(K) = 0$. But by the previous implication

$$\dim_K \Omega_{K_0}(K) = \operatorname{trdeg}_{K_0} K$$

Thus $\operatorname{trdeg}_{K_0} K = 0$, showing that $k(U) \subset K$ is algebraic. Now by Proposition2.4(a) there exists a subset $U' \subset U$ with U' a transcendence basis of K over k. Hence by the previous implication, $d(U')$ is a K-generating set of $\Omega_k(K)$. Since $\{dx_i\}$ is a K-basis of $\Omega_k(K)$ it follows that $U' = U$. □

Corollary 2.8. *If $k \subset K$ is any field extension, then*

$$\dim_K \Omega_k(K) = \operatorname{trdeg}_k K$$

3. Ramifications and singular locus

We assume throughout that k is a field of characteristic zero. The results of this section hold also in characteristic p if the field k is perfect.

Recall that a *coefficient field* for any local ring $(R,\mathfrak{m})$ is a subfield of R that maps isomorphically to the residue class field $R/\mathfrak{m}$. Furthermore, recall that any complete local ring (in characteristic zero) has a coefficient field. Let $\nu(\cdot)$ denote the minimal number of generators of a finite R-module. Recall that the embedding dimension of a Noetherian local ring is the minimal number of generators of its maximal ideal,

$$\operatorname{edim} R = \nu(\mathfrak{m}).$$

Theorem 3.1. *Let $(R,\mathfrak{m})$ be a local k-algebra essentially of finite type with $K = R/\mathfrak{m}$. Then*

$$\nu(\Omega_k(R)) = \operatorname{edim} R + \operatorname{trdeg}_k K$$

Proof. The ring $R/\mathfrak{m}^2$ is complete local. Thus it has a coefficient field containing k. Equivalently, the natural map $\pi : R/\mathfrak{m}^2 \longrightarrow K$ has a right inverse as a map of k-algebras. Thus by Proposition 2.3 we have an exact sequence

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow K \otimes_R \Omega_k(R) \longrightarrow \Omega_k(K) \longrightarrow 0$$

Now use Nakayamas Lemma and the fact that $\dim_K \Omega_k(K) = \operatorname{trdeg}_k(K)$. $\square$

Corollary 3.2 (Berger-Kunz)**.** *Let $(A,\mathfrak{n}) \longrightarrow (R,\mathfrak{m})$ be a local map making R an A-algebra essentially of finite type. Then*

$$\nu(\Omega_A(R)) = \nu(\mathfrak{m}/R\mathfrak{n}) + \operatorname{trdeg}_k K$$

Proof. Replacing $A \longrightarrow R$ by $k = A/\mathfrak{n} \longrightarrow R/R\mathfrak{n}$ we are in the situation of the previous theorem. Notice that (use Remark 1.8(b), or homework 2, or the conormal sequence)

$$\Omega_k(R/R\mathfrak{n}) = \Omega_A(R/R\mathfrak{n}) \cong \Omega_A(R)/Rd(\mathfrak{n}) + \mathfrak{n}\Omega_A(R) = \Omega_A(R)/\mathfrak{n}\Omega_A(R)$$

which shows that $\nu(\Omega_k(R/R\mathfrak{n})) = \nu(\Omega_A(R))$. $\square$

Let $(A,\mathfrak{n}) \longrightarrow (R,\mathfrak{m})$ be a local map making R an A-algebra essentially of finite type and $k \subset K$ be the extension of the residue fields. Then R is called *unramified* over A if $\mathfrak{m} = R\mathfrak{n}$ and $k \subset K$ is separable algebraic. (In our case, since the characteristic is zero, every extension is separable). Otherwise R is called *ramified* over A.

For any A-algebra R essentially of finite type we call

$$\operatorname{Ram}(R/A) = \{\mathfrak{q} \in \operatorname{Spec}(R) \mid R_{\mathfrak{q}} \text{ ramified over } A_{\mathfrak{q}\cap A}\}$$

the *ramification locus* of R over A.

Corollary 3.3. *Let $A \longrightarrow R$ be a local map making R an A-algebra essentially of finite type. Then R is unramified $\Longleftrightarrow \Omega_A(R) = 0$.*

Recall (see homework 7) Let $R^s \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$, then $\mathrm{Fitt}_i(M) = I_{n-i}(\varphi)$ is an increasing chain of ideals uniquely determined by M,

$$V(\mathrm{Fitt}_i(M)) = \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \nu(M_\mathfrak{q}) > i\},$$

hence

$$V(\mathrm{Fitt}_0(M)) = \mathrm{Supp}(M) = V(\mathrm{ann}(M))$$

Definition 3.4. If R is an algebra essentially of finite type over a Noetherian ring A and $M = \Omega_A(R)$ then M is finitely presented and the matrix φ can be obtained from a Jacobian matrix (see Corollary 1.12) One calls

$$\mathfrak{d}_K(R/A) = \mathrm{Fitt}_0(\Omega_A(R))$$

the *Kähler different* of R over A.

Corollary 3.5. *If R is an algebra essentially of finite type over a Noetherian ring A, then*

$$\mathrm{Ram}(R/A) = \mathrm{Supp}(\Omega_A(R)) = V(\mathfrak{d}_K(R/A))$$

In particular $\mathrm{Ram}(R/A)$ *is a closed subset of* $\mathrm{Spec}(R)$.

Proof. For $\mathfrak{q} \in \mathrm{Spec}(R)$ and $\mathfrak{p} = \mathfrak{q} \cap A$, we have $\Omega_{A_\mathfrak{p}}(R_\mathfrak{q}) \cong \Omega_A(R)_\mathfrak{q}$ by Proposition 1.14. Now use Corollary 3.3. □

Examples 3.6.

(1) Let $A = \mathbb{Z} \subset R = \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$. Here $\mathfrak{d}_K(R/A) = 2R$, hence

$$\mathrm{Ram}(R/A) = V(2R) = \{(1+i)R\}$$

Indeed, recall that the nonzero prime ideals of R are generated by prime elements of R which in turn arise in the following way, where p is any prime number and a,b are integers
  - (a) $p \equiv 3 \mod 4 \Longrightarrow p$ is a prime element in R
  - (b) $p \equiv 1 \mod 4 \Longrightarrow p = a^2+b^2 = (a+ib)(a-ib)$ with $a+ib \not\sim a-ib$ primes in R
  - (c) $p = 2 \Longrightarrow p \sim (1+i)^2$ with $1+i$ prime in R

(2) Let $A = k[x] \subset R = k[x,y]/(x-(y-1)^2)$ then we have a branch at the vertex of the parabola. Indeed if we project to the x-axis each point correspond to two separate points in the parabola but not the origin. Notice that $\mathfrak{d}_K(R/A) = 2(y-1)R$, which correspond to the origin of the parabola.

Recall Let R be a Noetherian local domain and M a finite R-module. Then $\nu(M) \geq \mathrm{rank}\, M$, and equality holds if and only if M is free.

Theorem 3.7 (Jacobian Criterion)**.** *Let $(R,\mathfrak{m})$ be a local k-algebra essentially of finite type with $K = R/\mathfrak{m}$. Write $D = \dim R + \mathrm{trdeg}_k K$. The following are equivalent.*

(1) R is regular

(2) $\Omega_k(R)$ is free of rank D

(3) $\operatorname{Fitt}_D(\Omega_k(R)) = R$

In this case $D = \operatorname{trdeg}_k L$ *where* $L = \operatorname{Quot}(R)$.

Proof. Assuming (1) we prove (2) and $D = \operatorname{trdeg}_k L$. By the dimension formula we have

$$\dim R = \operatorname{trdeg}_k L - \operatorname{trdeg}_k K$$

since R is a domain. In particular, $D = \operatorname{trdeg}_k L$.

$$\begin{aligned}
\operatorname{rank}\Omega_k(R) &\leq \nu(\Omega_k(R)) \\
&= \operatorname{edim} R + \operatorname{trdeg}_k K \qquad \text{by 3.1} \\
&= \dim R + \operatorname{trdeg}_k K \qquad \text{since } R \text{ is regular} \\
&= D = \operatorname{trdeg}_k L \\
&= \dim_L \Omega_k(L) \qquad \text{by 2.8} \\
&= \dim_L L \otimes_R \Omega_k(R) \qquad \text{by 1.11} \\
&= \operatorname{rank}\Omega_k(R)
\end{aligned}$$

(2) $\Longrightarrow$ (3) is clear

(3) $\Longrightarrow$ (1) Since $\operatorname{Fitt}_D(\Omega_k(R)) = R$, $\nu(\Omega_k(R)) \leq D$. But $\nu(\Omega_k(R)) = \operatorname{edim} R + \operatorname{trdeg}_k K$ by Theorem 3.1, and $D = \dim R + \operatorname{trdeg}_k K$. Thus $\operatorname{edim} R \leq \dim R$, which forces R to be regular. □

The equivalent statement of the following theorem in characteristic p require the ring R to be reduced.

Theorem 3.8. *Let R be a local k-algebra essentially of finite type. Then R is regular* $\Longleftrightarrow \Omega_k(R)$ *is free.*

Proof. Let $k \subset K$ be the residue field extension and $D = \dim R + \operatorname{trdeg}_k K$. By the Jacobian criterion we have to show that if $\Omega_k(R) \cong R^r$ then $r = D$. Let $\mathfrak{p}$ be a minimal prime of R with $\dim R = \dim R/\mathfrak{p}$. Now $\Omega_k(R_\mathfrak{p}) \cong R_\mathfrak{p}^r$ by Proposition 1.11. Also

$$\dim R_\mathfrak{p} + \operatorname{trdeg}_k K(\mathfrak{p}) = 0 + \operatorname{trdeg}_k R/\mathfrak{p} = \dim R/\mathfrak{p} + \operatorname{trdeg}_k K = \dim R + \operatorname{trdeg}_k K = D$$

Thus we may replace R by $R_\mathfrak{p}$ to assume that $(R, \mathfrak{m})$ is Artinian. Now

$$\dim_K \Omega_k(K) = \operatorname{trdeg}_k K = \dim R + \operatorname{trdeg}_k K = D$$

Now the conormal sequence gives

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} K \otimes_R \Omega_k(R) \cong K^r \longrightarrow \Omega_k(K) \cong K^D \longrightarrow 0$$

and we need to show that $\delta = 0$. Suppose that $\delta \neq 0$. Then $\delta(f) = 1 \otimes df \neq 0$ for some f in $\mathfrak{m}$. Hence df is part of a minimal generating set of $\Omega_k(R)$, hence pat of a basis as $\Omega_k(R)$ is free. Therefore $\operatorname{ann}_R(df) = 0$. Since $\mathfrak{m}$ is nilpotent there exists an $s > 0$ such that $f^s = 0$. Let s be minimal. Then $f^{s-1} \neq 0$ and since the characteristic is zero we get $sf^{s-1} \neq 0$ in R. But $sf^{s-1}df = df^s = 0$, which is impossible as $\operatorname{ann}_R(df) = 0$. □

Definition 3.9. Let k be a field, W a multiplicative subset of the polynomial ring $k[x_1,\ldots,x_n]$, I an ideal in $W^{-1}k[x_1,\ldots,x_n]$, and assume that every minimal prime of I has the same height g. Set $R=W^{-1}k[x_1,\ldots,x_n]/I$ and $D=n-g$. Then

$$J(R/k)=\mathrm{Fitt}_D(\Omega_k(R))$$

is called the *Jacobian ideal* of R over k.

Remark 3.10. We use the assumption of the Definition 3.9.

(1) For every $\mathfrak{p}\in\mathrm{Spec}(R)$, $\dim R_{\mathfrak{p}}+\mathrm{trdeg}_k K(\mathfrak{p})=D$. In particular, the integer D only depends on $k\subset R$, does not change when passing to a non zero ring of fractions, and coincides with the one of the Jacobian criterion if R is local.

(2) $J(R/k)$ only depends on $k\subset R$, and for $V\subset R$ a multiplicatively closed subset one has

$$J(V^{-1}R/k)=V^{-1}J(R/k)$$

Proof. (a) Let Q be the preimage of $\mathfrak{p}$ in $S=W^{-1}k[x_1,\ldots,x_n]$. Thus $R_{\mathfrak{p}}\cong S_Q/I_Q$ and $K(\mathfrak{p})=K(Q)$. Notice $\mathrm{ht}\, I_Q=\mathrm{ht}\, I=g$ by our assumption on I. Furthermore

$$\begin{aligned}\dim R_{\mathfrak{p}}+\mathrm{trdeg}_k K(\mathfrak{p}) &= \dim S_Q/I_Q+\mathrm{trdeg}_k K(Q)\\ &= \dim S_Q-\mathrm{ht}\, I_Q+\mathrm{trdeg}_k K(Q)\\ &= \mathrm{trdeg}_k S-\mathrm{ht}\, I_Q\\ &= n-g=D\end{aligned}$$

(b) Follows from part (a) and because the module of differentials localizes (see Proposition 1.11). □

Let R be a Noetherian ring. Then

$$\mathrm{Sing}(R)=\{\mathfrak{q}\in\mathrm{Spec}(R)\,|\,R_{\mathfrak{q}}\text{ not regular}\}$$

is called the *singular locus* of R.

Corollary 3.11. *Retain the assumption of Definition 3.9 . Then*

$$\mathrm{Sing}(R)=V(J(R/k))$$

In particular $\mathrm{Sing}(R)$ *is a closed subset of* $\mathrm{Spec}(R)$.

Proof. It follows easily from the Jacobian criterion and the above remark. □

Example 3.12. Let $R=\mathbb{C}[x,y,z]/(x^2-yz)$. Then $\Omega_{\mathbb{C}}(R)$ is presented by the Jacobian matrix $(2x,-y,-z)$, and thus $\mathrm{Fitt}_D(\Omega_{\mathbb{C}}(R)\cong(x,y,z)$. Thus by the Jacobian criterion, $R_{\mathfrak{p}}$ is a regular ring for all primes $\mathfrak{p}$ not containing (x,y,z). Thus R is regular on the punctured spectrum and the variety corresponding to R is smooth.

4. EVOLUTIONS AND REDUCTIONS

In this section we will introduce two kinds of dependence of ideals: *differential dependence* and *integral dependence*. Differential dependence is essentially equivalent to the notion of *evolution* of algebras essentially of finite type. The latter arises naturally in the study of Hecke algebras, as in the work of Wiles, Taylor-Wiles and Flach related to the proof of Fermat's Last Theorem [11, 12]. The notion of integral dependence is ubiquitous in commutative algebra and has played a crucial role in number theory and algebraic geometry since the nineteenth century.

Assume that $R = k[\![x_1,\ldots,x_n]\!]$ is either a power series ring or $R = k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ the localization at the irrelevant maximal ideal of a polynomial ring over a field k of characteristic zero. Let $J \subset I$ two proper ideals of R. Consider the projection

$$B = R/J \twoheadrightarrow A = R/I$$

We say that I is *differentially dependent* on J, or equivalently, B is an *evolution* of A if

$$\Omega_k(B) \otimes_B A \overset{\sim}{\twoheadrightarrow} \Omega_k(A)$$

The evolution is called *trivial* if $B \overset{\sim}{\twoheadrightarrow} A$ is already an isomorphism.

The definition of integral dependence of ideals takes place in the environment of several graded algebras attached to I, among which we single out the *Rees algebra* $\mathcal{R}(I)$, the *extended Rees algebra* $R[It,t^{-1}]$, the *associated graded ring* $\mathcal{G}(I)$, and the *special fiber ring* $\mathcal{F}(I)$. These algebras play a crucial role in the birational study of algebraic varieties, particularly in the process of desingularization. Indeed, they are usually referred to as *blowup algebras*, since they are the algebraic realizations of blowing up $\operatorname{Spec}(R)$ along $V(I)$. The Rees ring corresponds to the blowup of $\operatorname{Spec}(R)$ along $V(I)$, the associated graded ring to the exceptional fiber of the blowup, and the special fiber ring to the fiber over the closed point. These algebras also describe images of rational maps, secant varieties, tangential varieties and Gauss images of embedded projective varieties.

In these notes we will make use of the Rees algebra, the extended Rees ring, and the special fiber ring. Let R be any Noetherian local ring and I an R-ideal. We define

$$\begin{array}{ccc} \mathcal{R}(I) = R[It] = \oplus_{i=0}^{\infty} I^i t^i & \hookrightarrow & R[It,t^{-1}] = \oplus_{i\in\mathbb{Z}} I^i t^i \subset R[t,t^{-1}] \\ \downarrow & & \\ \mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = \oplus_{i=0}^{\infty} I^i/\mathfrak{m}I^i & & \end{array}$$

We say that I is *integral over* J, or equivalently, J is a *reduction* of I, if

$$\forall x \in I \qquad x^n + a_1x^{n-1} + \ldots + a_n = 0 \quad \text{with } a_j \in J^j$$
$$\iff \quad \mathcal{R}(J) \subset \mathcal{R}(I) \ \text{ is integral or module finite}$$
$$\iff \quad I^{n+1} = JI^n \qquad \text{for some } \ n \geq 0.$$

The second equivalence is easy to check since

$$\mathcal{R}(J) \subset \mathcal{R}(I) \quad \text{is integral or module finite}$$

$\Longleftrightarrow$ every element of It satisfies an equation of integrality over $R[Jt]$ that we may assume to respect the grading

$\Longleftrightarrow \forall xt \in It \quad \exists n \in \mathbb{N}$ and $a_i t^i \in R[Jt]_i = J^i t^i$ with $(xt)^n + (a_1 t)(xt)^{n-1} + \ldots + a_n t^n = 0$

$\Longleftrightarrow \forall x \in I$ is integral over J

Example 4.1. If $H = (g_1, \ldots, g_s)$ is an R ideal, then $(g_1^i, \ldots, g_s^i)$ is a reduction of H^i.

If $I = J + RU$ and every element of U is integral over J, then I is integral over J. Thus the *integral closure* of J

$$\overline{J} = \{x \in R \mid x \text{ integral over } J\}$$

is an ideal containing J. We say that J is *integrally closed* if $\overline{J} = J$; as integral dependence is transitive, the integral closure of J is integrally closed. Thus the integral closure is the unique largest ideal integral over J.

Remark 4.2. (1) $\overline{J} \subset \sqrt{J}$; in particular every radical ideal is integrally closed.

(2) Every principal ideal in a normal domain is integrally closed.

Now we assume again that $R = k[[x_1, \ldots, x_n]]$ is either a power series ring or $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ the localization at the irrelevant maximal ideal of a polynomial ring over a field k of characteristic zero. Let $J \subset I$ be two proper ideals of R. The two notions of dependence are related

Theorem 4.3 (Scheja-Storch [7])**.** *Differential dependence $\Longrightarrow$ integral dependence*

The next example shows how non-trivial evolutions may arise.

Example 4.4. *Let $f \in \mathfrak{m}$ and let J be the Jacobian ideal of f, that is*

$$J = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset I = (J, f)$$

Then I is differentially dependent on J, that is $B = R/J$ is an evolution of $A = R/(J, f)$. In particular, by Theorem 4.3, I is integral over J, that is

$$f \in \overline{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}$$

Proof. Using the conormal sequence (see 2.1)

$$H/H^2 \xrightarrow{\delta} \Omega_k(B) \otimes_B A \xrightarrow{\sim} \Omega_k(A) \longrightarrow 0$$

where $H = I/J = (J, f)/J$ is the kernel of the epimorphism $B \longrightarrow A$, it is enough to observe that $\delta(H) = 0$. This is obvious because $d(f + J) \in J\Omega_k(B) = 0$. ☐

Remark 4.5. Using the valuative criterion for integral closure of ideals, that is $r \in R$ is in $\bar{I} \iff r \in IV$ for all valuation rings V between R and $\mathrm{Quot}(R)$, we obtain a strengthening of the last conclusion in the example:

$$f \in \overline{\mathfrak{m}\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)}.$$

One can easily show this by using Cohen structure theorem and the chain rule (see [10, 7.1.5]). An important open question is whether

$$f \in \mathfrak{m}\overline{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)},$$

which is related to the the existence of non-trivial evolutions over the complex numbers. Indeed, as we will see below, a domain $A = R/\mathfrak{p}$ has a non-trivial evolution if and only if there is a minimal generator f of $\mathfrak{p}$ such that $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \subset \mathfrak{p}$, that is there exists a minimal generator f of $\mathfrak{p}$ such that $f \in \mathfrak{p}^{(2)}$ (see Homework 17).

If f is quasi-homogeneous the above example does not provide an example of a non-trivial evolutions, as by the Euler formula, $f \in J = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \Longrightarrow I = J$. More is true, indeed,

$$f \in J = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \iff f \text{ is quasi-homogenous}$$

The forward implication (due to Saito) $\Longrightarrow$ is much harder to prove, but it gives a very important criterion to check if an element is quasi-homogenous, (that is if there exists a change of coordinates that renders f quasi-homogenous).

Conjecture 4.6 (Eisenbud-Mazur [2]). *Assume in addition that A is a domain. Every evolution of A is an isomorphism.*

The above conjecture is known to hold in few cases, for instance, if I is a complete intersection, an almost complete intersection, or I is *licci*, etc.

To study the above conjecture we need to look at the conormal sequences (see 2.1) for A and B:

$$J/J^2 \xrightarrow{\delta} \Omega_k(R) \otimes_R B \longrightarrow \Omega_k(B) \longrightarrow 0$$

$$I/I^2 \xrightarrow{\delta} \Omega_k(R) \otimes_R A \longrightarrow \Omega_k(A) \longrightarrow 0$$

From this we get the following diagram, where the first sequence is obtained from the first sequence above after tensoring with A, and the second is obtained from the second sequence above and Homework 11

(2)
$$\begin{array}{ccccccccc} & & J/J^2 & \longrightarrow & \Omega_k(R) \otimes_R A & \longrightarrow & \Omega_k(B) \otimes_B A & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & I/I^{(2)} & \longrightarrow & \Omega_k(R) \otimes_R A & \longrightarrow & \Omega_k(A) & \longrightarrow & 0 \end{array}$$

Now the Snake Lemma shows that $B \longrightarrow A$ is an evolution of $A \iff I = J + I^{(2)}$. Thus we obtain:

Theorem 4.7 (Eisenbud-Mazur [2]). *Assume that is A a domain. Every evolution of A is an isomorphism $\Longleftrightarrow I^{(2)} \subset \mathfrak{m}I$.*

By a theorem of Huneke we know that the previous inclusion holds if $\mathfrak{m}I$ is integrally closed. So an important question is when are products of integrally closed ideals integrally closed? (Note that I is integrally closed since it is a prime ideal and therefore radical.)

Theorem 4.8 (Huneke). $I^{(2)} \subset \overline{\mathfrak{m}I}$.

Proof. From the bottom sequence of diagram (2), we observe that $f \in I^{(2)}$ implies $J = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \subset I$. (Alternatively, by definition $f \in I^{(2)}$ if and only if there exists z a non zero divisor I such that $zf \in I^2$. After taking derivatives, this shows that $J = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \subset I$.) Therefore

$$f \in \overline{\mathfrak{m}J} \subset \overline{\mathfrak{m}I}$$

□

Now we circle back to integral dependence and reductions. In this part of the section R is simply a Noetherian ring and there is no assumption on the characteristic. A reduction of an ideal is *minimal* if it is minimal with respect to inclusion. In a Noetherian local ring with infinite residue field minimal reductions always exist. The minimal number of generators of any minimal reduction of I is an invariant of I called the *analytic spread* of I, $\ell(I)$ (see Theorem 4.12). It turns out that the *analytic spread* is the Krull dimension of the special fiber ring of I

$$\ell(I) = \dim \mathcal{F}(I).$$

Indeed, we will show that minimal reductions arise from Noether normalizations of $\mathcal{F}(I)$ (see Theorem 4.12). We will see that the special fiber ring of a minimal reduction $\mathcal{F}(J)$ is a polynomial ring (see Theorem 4.12). We also deduce the following inequalities (see Corollary 4.13)

$$\operatorname{ht} I \leq \ell(I) \leq \min\{\nu(I), \dim R\}$$

In particular, every minimal reduction of an $\mathfrak{m}$-primary ideal is minimally generated by $\dim R$ elements and, if R is Cohen-Macaulay it is generated by a regular sequence.

Set $\ell(I) = \dim \mathcal{F}(I)$.

Proposition 4.9. *Let $A \subset B$ be a homogenous inclusion of Noetherian standard graded rings with $A_0 = B_0 = R$, and let $r \geq 0$ be an integer. The following are equivalent*

(1) B is a finite A-module, generated by homogenous elements of degree $\leq r$
(2) $A \subset B$ is integral
(3) $B_n = A_{n-r}B_r$ for all $n \geq r$
(4) $A_+B \subset B_+$ is integral, that is A_+B is a reduction of B_+
(5) $\sqrt{A_+B} = \sqrt{B_+}$, with $B_+^{r+1} \subset A_+B$.

If moreover $(R, \mathfrak{m})$ is local and $-'$ denotes images in $B' = B/\mathfrak{m}B$ then the above are equivalent to

(6) *$B/\mathfrak{m}B$ is a finite $A/\mathfrak{m}A$- module, generated by homogenous elements of degree $\leq r$*
(7) *$A' \subset B'$ is integral*
(8) *$B'_n = A'_{n-r}B'_r$ for all $n \geq r$*
(9) *$A'_+B' \subset B'_+$ is integral*
(10) *$\sqrt{A'_+B'} = B'_+$, with $B'^{r+1}_+ \subset A'_+B'$*
(11) $\mathrm{ht} A'_+B' = \dim B'$

Proof. Notice that since B Noetherian, B is a finitely generated R-algebra and all B_i are finite R-modules.

(1) $\Longleftrightarrow$ (2) Clear since B is a finitely generated A-algebra.

(1) $\Longleftrightarrow$ (3) $B_n = A_{n-r}B_r$ for all $n \geq r \Longleftrightarrow B_n \subset B_rA$ for all $n \geq r \Longleftrightarrow B = \sum_{i=0}^{r} B_iA \Longleftrightarrow B$ is generated as A-module by finitely many elements of degree $\leq r$ (recall that B_i are finite R-modules).

(3) $\Longleftrightarrow$ (4) $B_n = A_{n-r}B_r$ for all $n \geq r \Longleftrightarrow B_nB = (A_{n-r}B)(B_rB)$ for all $n \geq r \Longleftrightarrow B^n_+ = (A_+B)^{n-r}B^r_+$ for all $n \geq r$ (recall A and B are standard graded).

(3) $\Longleftrightarrow$ (5) $B^{r+1}_+ \subset A_+B \Longleftrightarrow B_{r+1} \subset A_+B \Longleftrightarrow B_{r+1} = A_1B_r \Longleftrightarrow B_n = A_{n-r}B_r$ for all $n \geq r$

The equivalence of (6)-(10) follows from the equivalence of (1)-(5) applied to $A' \subset B'$.

(3) $\Longleftrightarrow$ (8) $B'_n = A'_{n-r}B'_r \Longleftrightarrow B_n = A_{n-r}B_r + \mathfrak{m}B_n \Longleftrightarrow B_n = A_{n-r}B_r$ where the last equivalence holds by Nakayama's lemma (recall B_n is a finite R-module and $(R, \mathfrak{m})$ is local).

(10) $\Longrightarrow$ (11) Since B' is a standard graded ring with $B'_0 = R/\mathfrak{m}$ a field, $\mathrm{ht} B'_+ = \dim B'$.

(11) $\Longrightarrow$ (10) If $\mathfrak{p}$ is a minimal prime of A'_+B', then $\mathfrak{p}$ is homogeneous, hence a homogeneous maximal ideal. Thus $\mathfrak{p} = B'_+$.

□

Corollary 4.10. *Let R be Noetherian ring and $J \subset I$ ideals of R. The following are equivalent*

(1) *I is integral over J*
(2) *J is reduction of I*
(3) *$\mathcal{R}(I)$ is a finite $\mathcal{R}(J)$-module*
(4) *$\mathcal{R}(I)_+ = It\mathcal{R}(I)$ is integral over $Jt\mathcal{R}(I)$*
(5) *$\sqrt{\mathcal{R}(I)_+} = \sqrt{Jt\mathcal{R}(I)}$.*

If moreover $(R, \mathfrak{m}, k)$ is local and $J' = (J + \mathfrak{m}I)/\mathfrak{m}I \subset \mathcal{F}(I)_1$, then the above are equivalent to

(6) *$\mathcal{F}(I)$ is integral over $k[J']$*
(7) *$\mathcal{F}(I)_+$ is integral over $J'\mathcal{F}(I)$*
(8) *$\sqrt{J'\mathcal{F}(I)} = \mathcal{F}(I)_+$*
(9) $\mathrm{ht} J'\mathcal{F}(I) = \ell(I)$

Lemma 4.11. *Let k be a an infinite field and $\mathcal{F}$ any Noetherian standard graded k-algebra of dimension ℓ. Then there exist linear forms $x_1, \ldots, x_\ell$ in $\mathcal{F}$ with $\mathcal{F}_+ = \sqrt{(x_1, \ldots, x_\ell)}$.*

Proof. The claim is clear for $\ell = 0$. Write $\operatorname{Min}(\mathcal{F}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. By induction on ℓ it suffices to show that if $\ell > 0$ then there exists $x \in \mathcal{F}_1$ with $x \notin \mathfrak{p}_i$ for all i. Since $\mathcal{F}_+ = \mathcal{F}_1\mathcal{F}$ and $0 < \ell = \operatorname{ht}\mathcal{F}_+$, $\mathcal{F}_1 \not\subset \mathfrak{p}_i$ for all i, hence $\mathfrak{p}_i \cap \mathcal{F}_1$ is strictly contained in $\mathcal{F}_1$ for all i. Thus the union of all $\mathfrak{p}_i \cap \mathcal{F}_1$ is strictly contained in $\mathcal{F}_1$ because a finite dimensional vector space over an infinite field is not a finite union of proper subspaces. □

Theorem 4.12. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with infinite residue field, I an R-ideal and J a reduction of I. Then*

(a) $\nu(J) \geq \ell(I)$

(b) $\nu(J) = \ell(I) \Longleftrightarrow$ *J is a minimal reduction of I*

(c) *If J is a minimal reduction and $\ell = \ell(I)$ then*

$$k[T_1, \dots, T_\ell] \xrightarrow{\sim} \mathcal{F}(J) \hookrightarrow \mathcal{F}(I)$$

In particular, the minimal generators of J are minimal generators of I

$$J \otimes_R k \hookrightarrow I \otimes_R k$$

(d) *I has a minimal reduction, and every reduction of I contains a minimal reduction of I*

Proof. Write $\mathcal{F} = \mathcal{F}(I)$ and $J' = (J + \mathfrak{m}I)/\mathfrak{m}I \subset \mathcal{F}_1$. Recall $\mathcal{F}$ is a standard graded k-algebra of dimension ℓ. Since $|k| = \infty$, by the previous lemma there are ℓ linear forms in $\mathcal{F}$ generating $\mathcal{F}_+$ up to radical. Also by Corollary 4.10

$$J \text{ is a reduction of } I \Longleftrightarrow \operatorname{ht} J'\mathcal{F} = \ell \Longleftrightarrow \sqrt{J'\mathcal{F}} = \mathcal{F}_+ .$$

Therefore

(a) $\nu(J) \geq \dim_k J' = \nu(J'\mathcal{F}) \geq \operatorname{ht} J'\mathcal{F} = \ell$, where the second inequality follows from Krull' s Altitude Theorem.

(b) and (d) J is a minimal reduction of I

$\Longleftrightarrow$ J' is minimal with $\operatorname{ht} J'\mathcal{F} = \ell$ and J is a minimal representative of its image J'

$\Longleftrightarrow$ $\operatorname{ht} J'\mathcal{F} = \ell, \quad \dim_k J' = \ell, \quad \nu(J) = \ell$

$\Longleftrightarrow$ J is a reduction of I and $\nu(J) = \ell$

(c)

$$k[T_1, \dots, T_\ell] \xrightarrow{\varphi} \mathcal{F}(J) \xrightarrow{\Psi} \mathcal{F}$$

with $\Psi \cdot \varphi$ module finite by the above Lemma 4.11. Hence injective since $\dim \mathcal{F}(J) = \dim \mathcal{F} = \ell$, hence φ is an isomorphism and Ψ is injective. □

Corollary 4.13. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and I a proper ideal. Then*

$$\operatorname{ht} I \leq \ell(I) \leq \min\{\nu(I), \dim R\} .$$

Proof. We may assume k is infinite. Let J be a minimal reduction of I, then

$$\operatorname{ht} I \overset{4.2}{=} \operatorname{ht} J \leq \nu(J) \overset{4.12(b)}{=} \ell(I) \overset{4.12(a)}{\leq} \nu(I)$$

where the first inequality is Krull altitude theorem. Finally, $\ell(I) \leq \dim R$ as the special fiber ring is an epimorphic image of the associated graded ring and the Krull dimension of the latter is $\dim R$. $\square$

5. Three differents and the Briançon-Skoda theorem

In the previous section we have seen that if $R = k[[x_1, \ldots, x_n]]$ is a power series over a field of characteristic zero and $f \in \mathfrak{m} = (x_1, \ldots, x_n)$, then

$$f \in \overline{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)} \subset \sqrt{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}.$$

In particular, there exists an integer k such that

$$f^k \in \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Mather's Question: *Is there a k that works uniformly for all f?*

The answer is yes and the bound is n, the Krull dimension of the ring R. It is given by the Briançon-Skoda theorem, which we will prove in this section. This famous theorem was first proved by analytic means. In the 1980's, Lipman-Sathaye (see [5]) gave an algebraic proof. Later Hochster-Huneke provided a vast generalization using the theory of tight closure (see [4, 1.5.5 and 4.1.5]). In this section we will only treat the case of an equimultiple ideal (that is, an ideal with analytic spread equal to its grade, equivalently, an ideal having a minimal reduction generated by a regular sequence) in a regular local algebra essentially of finite type over a field of characteristic zero (the same statement with the same proof holds if k is a perfect field).

Throughout, we assume that k is a field of characteristic zero. Our proof, along the lines of [5], will consists of two important steps together with the Jacobian criterion. The first one says the Jacobian ideal $J(R/k)$ is contained in the *conductor* $R : \overline{R}$ of R.

Theorem 5.1. *Let $(R, \mathfrak{m}, k)$ be a reduced, equidimensional, k-algebra essentially of finite type. Then*

$$J(R/k) \subset R : \overline{R}.$$

The second step is a version of the Briançon-Skoda theorem that works in non-regular rings.

Theorem 5.2. *Let $(R, \mathfrak{m}, k)$ be a reduced Cohen–Macaulay k-algebra essentially of finite type, and let I be an equimultiple ideal of height $g > 0$. Then for every integer n,*

$$J(R/k)\,\overline{I^{n+g-1}} \subset I^n.$$

Let $A \longrightarrow R$ be an algebra. To prove the first theorem we have to introduce three differents, the *Dedekind different*, the *Kähler different* (which we already introduced in Definition 3.4), and the *Noether different*. Afterwords we have to prove the following relations among them

$$\mathfrak{d}_K(R/A) \subset \mathfrak{d}_N(R/A) \subset \mathfrak{d}_D(R/A) \subset R : \overline{R}.$$

If R is a complete intersection the first three containments become equalities. This leads to the important fact that the socle of a complete intersection is given by the Jacobian ideal (see Homework 21 and [3]).

We prove the first theorem when R is complete. In this setting

$$R = k[\![x_1, \ldots, x_n]\!]/J$$

for some ideal $J \subset k[\![x_1, \ldots, x_n]\!]$ (see homework 15 for the definition of the module of differential in the complete case). We first introduce the three differents.

Let A be a Noether normalization of R. Recall that a Noether normalization of R is power series ring

$$A = k[\![y_1, \ldots, y_d]\!] \subset R$$

so that R is a finitely generated A-module. This is equivalent to $y_1, \ldots, y_d$ being a system of parameters of R.

Recall that the Kähler different of R over A is the zero-th Fitting ideal of the module of differentials

$$\mathfrak{d}_K(R/A) = \mathrm{Fitt}_0(\Omega_A(R)).$$

Using the third construction of the module of differential (see Theorem 1.10) we obtain the *Noether different* of R over A. Recall the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow R \otimes_A R \overset{\mu}{\longrightarrow} R \longrightarrow 0$$

where μ is the multiplication map. Then the Noether different is the image in R of the annihilator of the ideal $\mathbb{D}$ of the enveloping algebra

$$\mathfrak{d}_N(R/A) = \mu(0 :_{R \otimes_A R} \mathbb{D}).$$

Write $K = \mathrm{Quot}(A)$ and $L = \mathrm{Quot}(R)$. Notice that since L is reduced, L is a finite product of fields L_i. Each finite extension

$$K \subset L_i$$

gives rise to a nonzero trace map $\mathrm{Tr}_i : L_i \longrightarrow K$, which provides a non zero element $\mathrm{Tr} \in \mathrm{Hom}_K(L, K)$. The L-module $\mathrm{Hom}_K(L, K)$ is cyclic, thus must be generated by Tr (recall that $L = \times L_i$ is a product of fields). If $\varphi \in \mathrm{Hom}_A(R, A)$, the map $\varphi \otimes_A K$ is in $\mathrm{Hom}_K(L, K)$. Thus we obtain an inclusion

$$\mathrm{Hom}_A(R, A) \subset \mathrm{Hom}_K(L, K) = L \cdot \mathrm{Tr},$$

which shows that the finitely generated R-module $\mathrm{Hom}_A(R, A)$ is a product of the trace map, Tr, by an R-submodule of L, $\mathfrak{C}_{R/A}$, called the *complementary Dedekind module* of R/A

$$\mathrm{Hom}_A(R, A) = \mathfrak{C}_{R/A} \cdot \mathrm{Tr}.$$

Notice that the complementary module is

$$\mathfrak{C}_{R/A} = \{z \in L \mid \operatorname{Tr}(z \cdot R) \subset A\}$$

One has $1 \in \mathfrak{C}_{R/A}$ as $\operatorname{Tr}(R) \subset A$. This follows because, $\operatorname{Tr}(R) \subset K = \operatorname{Quot}(A)$, $\operatorname{Tr}(R)$ is integral over A, and A is integrally closed. In particular, we obtain the inclusion

$$R \subset \mathfrak{C}_{R/A}\,.$$

Let $\overline{R}$ denotes the integral closure of R in its total ring of fractions. Let

$$\mathfrak{C}_{\overline{R}/A} = \{z \in L \mid \operatorname{Tr}(z \cdot \overline{R}) \subset A\}$$

Again $1 \in \mathfrak{C}_{\overline{R}/A}$. Now that definition of the complementary module gives

$$R \subset \overline{R} \subset \mathfrak{C}_{\overline{R}/A} \subset \mathfrak{C}_{R/A}\,.$$

Finally we are ready to define the third different, the *Dedekind different* of R over A, $\mathfrak{d}_D(R/A)$, as the inverse of the Dedekind complementary module

$$\mathfrak{d}_D(R/A) = R :_L \mathfrak{C}_{R/A}\,.$$

Since $\overline{R} \subset \mathfrak{C}_{R/A}$ the Dedekind different is an R-ideal that is contained in the conductor $R : \overline{R}$ of R

$$\mathfrak{d}_D(R/A) = R :_L \mathfrak{C}_{R/A} \subset R :_L \overline{R}\,.$$

Theorem 5.3. *If $(R, \mathfrak{m}, k)$ is a complete reduced and equidimensional Noetherian k-algebra, and A is a Noether normalization of R, then*

$$\mathfrak{d}_K(R/A) \subset \mathfrak{d}_N(R/A) \subset \mathfrak{d}_D(R/A) \subset R : \overline{R}\,.$$

Proof. We need to show the first two inclusions

The first inclusion $\mathfrak{d}_K(R/A) \subset \mathfrak{d}_N(R/A)$: Consider the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow R \otimes_A R \xrightarrow{\mu} R \longrightarrow 0\,.$$

The zeroth Fitting ideal of a module is always contained in the annihilator of the module (see homework 7), thus

$$\operatorname{Fitt}_0(\mathbb{D}) \subset 0 :_{R \otimes_A R} \mathbb{D}\,.$$

Now we apply μ:

$$\begin{array}{ccc} \mu(\operatorname{Fitt}_0(\mathbb{D})) & \subset & \mu(0 :_{R\otimes_A R} \mathbb{D}) \\ \text{claim}\,\Big\| & & \Big\| \\ \mathfrak{d}_K(R/A) & \subset & \mathfrak{d}_N(R/A) \end{array}$$

We claim that the left hand side is $\mathfrak{d}_K(R/A)$. Indeed $\mu(\operatorname{Fitt}_0(\mathbb{D}))$ is the image of $\operatorname{Fitt}_0(\mathbb{D})$ in R so we can think of the application of μ as a base change. But Fitting ideals are compatible with base change. But $R \cong R \otimes_A R/\mathbb{D}$, thus we obtain, using the third definition of the module of differentials (see Definition 1.10),

$$\begin{aligned}\mu(\mathrm{Fitt}_0(\mathbb{D})) = \mathrm{Fitt}_0(\mathbb{D} \otimes_{R\otimes_A R} R) &= \mathrm{Fitt}_0(\mathbb{D} \otimes_{R\otimes_A R} (R \otimes_A R/\mathbb{D})) \\ &= \mathrm{Fitt}_0(\mathbb{D}/\mathbb{D}^2) = \mathrm{Fitt}_0(\Omega_A(R)) = \mathfrak{d}_K(R/A)\end{aligned}$$

The second inclusion $\mathfrak{d}_N(R/A) \subset \mathfrak{d}_D(R/A)$: (we show it only in the domain case) By definition $\mathfrak{d}_D(R/A) = R :_L \mathfrak{C}_{R/A}$. Thus it will be enough to show that

$$\mathfrak{d}_N(R/A) \cdot \mathfrak{C}_{R/A} \subset R$$

The Noether different is the annihilator of the ideal $\mathbb{D}$ defined by the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow R \otimes_A R \xrightarrow{\mu} R \longrightarrow 0\,.$$

Now in the enveloping algebra $R \otimes_A R$ we think of one copy of R as $R = A[\underline{X}]/P$, where $\underline{X}$ is a finite set of variables. We write $\underline{x}$ for the image of $\underline{X}$ in R. Now

$$R \otimes_A R = R[\underline{X}]/RP\,.$$

With this identification, the ideal $\mathbb{D}$ is generated by the images in $R \otimes_A R$ of $\underline{X} - \underline{x}$ and the Noether different is the image in R of the polynomials in $R[\underline{X}]$ that send $\underline{X} - \underline{x}$ into RP, that is

$$\mathfrak{d}_N(R/A) = \{G(\underline{x}) \in R \mid G(\underline{X}) \cdot (X_i - x_i) \in RP \quad \forall i\}\,.$$

We consider the distinct $K = \mathrm{Quot}(A)$-embeddings δ_j of L into its algebraic closure $\overline{L}$ and we extend them to $\delta_j : L[\underline{X}] \longrightarrow \overline{L}[\underline{X}]$ coefficientwise. Recall that the trace map is defined as the sum of such embedddings: $\mathrm{Tr}(b) = \sum \delta_j(b)$ for all $b \in L$.

Let $G(\underline{x}) \in \mathfrak{d}_N(R/A)$ we claim that

$$\delta_j(G)(\underline{x}) = 0 \qquad \text{if } \ \delta_j \neq \mathrm{id}\,.$$

Indeed, $\delta_j(G(\underline{X}) \cdot (X_i - x_i)) = \delta_j(G)(\underline{X})(X_i - \delta_j(x_i)) \in RP$, where the first equality holds because δ_j is extended to $L[\underline{X}]$ coefficientwise. When we apply μ, this element is mapped to zero since P is in the kernel of μ and we obtain

$$\delta_j(G)(\underline{x})(x_i - \delta_j(x_i)) = 0\,.$$

But if $\delta_j \neq \mathrm{id}$ then $x_i - \delta_j(x_i) \neq 0$ for some i, hence $\delta_j(G(\underline{x})) = 0$. This establish the claim.

Hence for every $z \in L$ the trace of $zG(\underline{x})$ is

$$\mathrm{Tr}(zG)(\underline{x}) = \sum \delta_j(z)\delta_j(G)(\underline{x}) = zG(\underline{x})\,.$$

Finally take $z \in \mathfrak{C}_{R/A} = \{z \in L \mid \mathrm{Tr}(z \cdot R) \subset A\}$. One has $zG(\underline{x}) = \mathrm{Tr}(zG)(\underline{x}) \subset A[\underline{x}] = R$. Notice that $\mathrm{Tr}(zG) \subset A$ by definition of the complementary module because each embedding, and henceforth the trace map, is extended to the polynomial ring coefficientwise and the coefficients of G are in R. Thus we have shown that for every $z \in \mathfrak{C}_{R/A}$ and every $G(\underline{x}) \in \mathfrak{d}_N(R/A)$, the product $zG(\underline{x})$ is in R, which establishes the claim. □

Theorem 5.4. *If $(R, \mathfrak{m}, k)$ is a complete Noetherian reduced and equidimensional k-algebra, then*

$$J(R/k) \subset R : \overline{R}.$$

Proof. The assertion follows from Theorem 5.3 once we have shown the following fact

Claim: $J(R/k) = \sum \mathfrak{d}_K(R/A)$,

where the sum is taken over all Noether normalizations A of R. Indeed by Theorem 5.3, each of the summands is contained in the conductor, but the conductor is an ideal and then it contains the whole sum and therefore the Jacobian ideal as well.

To show the claim, notice that we can choose the presentation of R

$$R = k[[x_1, \ldots, x_n]]/J$$

in such a way that any subset of $x_1, \ldots, x_n$ consisting of $d = \dim R$ elements gives rise to a Noether normalization A of R. Thus

$$J(R/k) = \mathrm{Fitt}_d(\Omega_k(R)) = \sum \mathrm{Fitt}_0(\Omega_A(R)) = \sum \mathfrak{d}_K(R/A)$$

as can easily seen by taking a presentation matrix of the module of differentials with respect to the chosen presentation. □

Theorem 5.5. [6, 3.1] *Let $(R, \mathfrak{m}, k)$ be a reduced Cohen–Macaulay local k-algebra essentially of finite type, and let I be an equimultiple ideal of height $g > 0$. Then for every integer n,*

$$J(R/k)\,\overline{I^{n+g-1}} \subset I^n.$$

Proof. Passing to a minimal reduction, we may suppose that I is generated by a regular sequence of length g. Let S be a finitely generated k–subalgebra of R so that $R = S_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}(S)$, and write $S = k[x_1, \ldots, x_e] = k[X_1, \ldots, X_e]/\mathfrak{a}$ with $\mathfrak{a} = (h_1, \ldots, h_t)$ an ideal of height c. Notice that S is reduced and equidimensional. Let $K = (f_1, \ldots, f_g)$ be an S–ideal with $K_{\mathfrak{p}} = I$, and consider the extended Rees ring $B = S[Kt, t^{-1}]$. Now B is a reduced and equidimensional affine k–algebra of dimension $e - c + 1$.

Let $\varphi\colon k[X_1, \ldots, X_e, T_1, \ldots, T_g, U] \longrightarrow B$ be the k–epimorphism mapping X_i to x_i, T_i to $f_i t$ and U to t^{-1}. Its kernel has height $c + g$ and contains the ideal $\mathfrak{b}$ generated by $\{h_i, T_jU - f_j | 1 \le i \le t, 1 \le j \le g\}$. Consider the Jacobian matrix of these generators,

$$\Theta = \left(\begin{array}{c|cccc} \dfrac{\partial h_i}{\partial X_j} & & 0 & & \\ \hline & U & & & T_1 \\ * & & \ddots & & \vdots \\ & & & U & T_g \end{array}\right).$$

Notice that $I_{c+g}(\Theta) \supset I_c(\left(\frac{\partial h_i}{\partial X_j}\right))U^{g-1}(T_1, \ldots, T_g)$. Applying φ we obtain $J(B/k) \supset I_{c+g}(\Theta)B \supset J(S/k)Kt^{-g+2}$. Thus by Theorem 5.4, $J(S/k)Kt^{-g+2}$ is contained in the conductor of B. Localizing at $\mathfrak{p}$ we see that $J(R/k)It^{-g+2}$ is in the conductor of the extended Rees ring $R[It,t^{-1}]$. Hence for every n, $J(R/k)I\overline{I^{n+g-1}} \subset I^{n+1}$, which yields

$$J(R/k)\overline{I^{n+g-1}} \subset I^{n+1} : I = I^n,$$

as $\mathrm{gr}_I(R)$ is Cohen-Macaulay being the associated graded ring of a regular sequence. □

Using the fact that in a regular ring the Jacobian ideal is trivial (see Theorem 3.7), we obtain

Corollary 5.6 (Briançon-Skoda)**.** *Let $(R,\mathfrak{m},k)$ be a regular local k-algebra essentially of finite type, and let I be an equimultiple ideal of height $g > 0$. Then for every integer n,*

$$\overline{I^{n+g-1}} \subset I^n.$$

The Briançon-Skoda theorem tells us that for every reduction J of I

$$\overline{I^g} \subset J \quad \Longrightarrow \quad \overline{I^g} \subset \cap J := \mathrm{core}(I)$$

where the intersection is taken over all minimal reductions of I.

6. Conjectures

In addition to the Eisenbud-Mazur conjecture, there are many other interesting and older conjectures concerning the module of differentials.

Setting 6.1. Assume k is a perfect field, R is a local k-algebra essentially of finite type that is reduced, K is the residue field, and $D = \dim R + \mathrm{trdeg}_k K$.

Conjecture 6.2 (Berger)**.** Suppose that $D = 1$. Then $\Omega_k(R)$ is torsion free if and only if R is regular. In particular $\Omega_k(R)$ is torsionfree if and only if is free.

The following conjecture is false in positive characteristic. It is essentially known, since it is known if the ring is R_2. It is open in the two dimensional case.

Conjecture 6.3 (Zariski-Lipman)**.** Suppose $\mathrm{char}(k) = 0$. Then $\mathrm{Der}_k(R)$ is free if and only if R is regular.

The following conjecture of Vasconcelos is wide open and suggests that the projective dimension is always 0,1, or infinite.

Conjecture 6.4 (Vasconcelos)**.** If $\Omega_k(R)$ has finite projective dimension over R, then R is a complete intersection (which is equivalent to $\mathrm{pd}_R\,\Omega_k(R) \le 1$, see Homework 11).

7. Exercises

(1) Let $R = A[x,y]/(xy)$. Compute $\Omega_A(R)$.

(2) Let $R = A[x,y,z]/(y^2 - x^2(z^2 - x))$. Compute $\Omega_A(R)$.

(3) **Base Change** Let $A \longrightarrow B$, $A \longrightarrow R$ be algebras. Prove that there is a $B \otimes_A R$-isomorphism

$$\Omega_B(B \otimes_A R) \cong B \otimes_A \Omega_A(R) \qquad \text{with} \quad d(b \otimes r) \mapsto b \otimes dr$$

(4) **Differentials make direct sums out of tensor products** Let $A \longrightarrow R_1$, $A \longrightarrow R_2$ be algebras, $T = R_1 \otimes R_2$. Prove that there is T-isomorphism

$$\Omega_A(T) \cong R_1 \otimes_A \Omega_A(R_2) \oplus R_2 \otimes_A \Omega_A(R_1) \qquad \text{with} \quad d(r_1 \otimes r_2) \mapsto r_1 \otimes dr_2 + r_2 \otimes dr_1$$

Deduce that for any (finite) collection of A-algebras R_i and $T = \otimes_A R_i$ the module of differential of T over A is

$$\Omega_A(T) \cong \oplus_i (\otimes_{j \neq i} R_j) \otimes_A \Omega_A(R_j)$$

(5) **Direct Products** Let $R_1, \ldots, R_n$ be A-algebras and $T = R_1 \times \ldots \times R_n$. Prove that there is a T-isomorphism

$$\Omega_A(T) \cong \oplus \Omega_A(R_i) \qquad \text{with} \quad d((r_i)) \mapsto (dr_i)$$

(6) **Colimits** Let $\{R_i, \psi_{ji}\}$ be a direct system of A-algebras. Write $\otimes_A R_i$ for the *restricted tensor product*, the A-submodule of the tensor product generated by the tensors almost all of whose factors are 1. This is an A-algebra. Let $\mathcal{J}$ be the ideal generated by $\{r_i - \psi_{ji}(r_i) \mid i \leq j, r_i \in R_i\}$. Then the A-algebra

$$\varinjlim R_i = \otimes_A R_i / \mathcal{J}$$

together with the natural A-algebra maps $\gamma_i : R_i \longrightarrow \varinjlim R_k$ is called the *direct limit* of the direct system $\{R_i, \psi_{ji}\}$. Notice $\gamma_i = \gamma_j \cdot \psi_{ji}$ for $i \leq j$. One has a universal property analogous the universal property of the usual direct limit.

For $i \leq j$ we have unique maps Ψ_{ji} that are linear over $T = \varinjlim R_i$

$$\begin{array}{ccc} T \otimes_{R_i} \Omega_A(R_i) & \xrightarrow{\Psi_{ji}} & T \otimes_{R_j} \Omega_A(R_j) \\ \uparrow{\scriptstyle d_{R_i/A}} & & \uparrow{\scriptstyle d_{R_j/A}} \\ R_i & \xrightarrow{\psi_{ji}} & R_j \end{array}$$

Now $\{T \otimes_{R_i} \Omega_A(R_i), \Psi_{ji}\}$ is a direct system of T-modules.

Prove that there is a T-isomorphism

$$\Omega_A(T) \cong \varinjlim T \otimes_{R_i} \Omega_A(R_i) \qquad \text{with} \quad d(\gamma_i(r_i)) \mapsto \alpha_i(1 \otimes dr_i)$$

(7) **Fitting Ideals** For an n by m matrix φ with entries in R write $I_t(\varphi)$ for the R-ideal generated by all t by t minors of φ (set $I_t(\varphi) = R$ for $t \le 0$ and $I_t(\varphi) = 0$ for $t > \min\{m,n\}$). Thinking of φ as an R-linear map

$$\varphi : R^m \longrightarrow R^n$$

set $M = \operatorname{coker}\varphi$ and define $F_i(M) = \operatorname{Fitt}_i(M) = I_{n-i}(\varphi)$. This ideal is called the *i-th Fitting ideal of M*. Show

(a) $F_i(M)$ only depends on i and M (but not on m,n, or φ).

(b) $(\operatorname{ann}(M))^n \subset F_0(M) \subset \operatorname{ann}(M)$

(c) If R is local, then $F_i(M) = R \Longleftrightarrow \nu(M) \le i$

(d) $V(F_i(M)) = \{\mathfrak{p} \in \operatorname{Spec}(R) \,|\, \nu_{R_\mathfrak{p}}(M_\mathfrak{p}) > i\}$

(8) Compute the singular locus of the Whitney umbrella $R = k[x,y,t]/(y^2 - x^2(t^2 - x))$.

(9) Use the assumption of Definition 3.9. Prove that

(a) $\operatorname{ht} J(R/k) \ge t \Longleftrightarrow R$ satisfies Serre's condition (R_{t-1})

(b) $\operatorname{grade} J(R/k) \ge t \Longleftrightarrow R$ satisfies (R_{t-1}) and (S_t)

(10) Use the assumption of Definition 3.9 and write $L = \operatorname{Quot}(R)$. Prove that

(a) R is reduced $\Longleftrightarrow \operatorname{grade} J(R/k) \ge 1 \Longleftrightarrow 0 :_R J(R/k) = 0$

(b) R is normal $\Longleftrightarrow \operatorname{grade} J(R/k) \ge 2 \Longleftrightarrow 0 :_R J(R/k) = 0$ and $R :_L J(R/k) = R$

(11) Let k be a perfect field and R a reduced local k-algebra essentially of finite type. Write $R = S/I$ where S is a regular local k-algebra essentially of finite type. Let W be the complement of the union of all associated primes of I and consider the symbolic square

$$I^{(2)} = S \cap (I^2 W^{-1} S)$$

Show that there is an exact sequence

$$0 \longrightarrow I/I^{(2)} \longrightarrow R \otimes_S \Omega_k(S) \longrightarrow \Omega_k(R) \longrightarrow 0$$

(12) Let S be a regular local k-algebra essentially of finite type and I an S-ideal. Show that $f \in I^{(2)} \Longleftrightarrow \partial(f) \in I$ for all $\partial \in \operatorname{Der}_k(S)$.

(13) Prove that the following are equivalent with the assumption of problem 11.

(a) I is a complete intersection

(b) I/I^2 is a free R-module

(c) $I/I^{(2)}$ is a free R-module

(d) $\operatorname{pd}_R \Omega_k(R) \le 1$

(14) Let k be a field with $\operatorname{char} k = 0$ and R a local k-algebra essentially of finite type. Show that if $\operatorname{Der}_k(R)$ is free then R is normal.

(15) **The complete case** ([1, Exercise 16.14]) In the complete case the module of differentials as we have defined it is not so useful. For example, if R is the localization of an affine ring over a field k at a maximal ideal and $\widehat{R}$ is its completion , then in general

$$\Omega_k(\widehat{R}) \neq \widehat{R} \otimes_R \Omega_K(R)$$

For most application it is the latter that is interesting. It turns out that this is the completion of $\Omega_k(\widehat{R})$, and that in general the completed module of differential is the right object to consider. Prove the following:

(a) Let $R = \mathbb{Q}[[x_1,\ldots,x_n]]$ be the ring of formal power series in n variables over the rational numbers. Show that the module of differentials $\Omega_{\mathbb{Q}}(R)$ is not a finitely generated R-module.

(b) Let $(R,\mathfrak{m})$ be a complete local ring with coefficient field k. Show that the completion

$$\widehat{\Omega_k(R)} = \varprojlim_n \Omega_k(R)/\mathfrak{m}^n\Omega_k(R)$$

may be identified with the inverse limits of the modules $\Omega_k(R/\mathfrak{m}^n)$. Show that

$$\widehat{\Omega_k(R)} = \Omega_k(R)/\cap_{j=1}^{\infty}\mathfrak{m}^j\Omega_k(R)$$

Show that the natural derivation

$$R \longrightarrow R/\mathfrak{m}^n \longrightarrow \Omega_k(R/\mathfrak{m}^n)$$

give rise to a derivation $\widehat{d} : R \longrightarrow wh\Omega_k(R)$, which may also be identified as the composite of the universal derivation $d : R \longrightarrow \Omega_k(R)$ and the projection map $\Omega_k(R) \longrightarrow \Omega_k(R)/\cap_{j=1}^{\infty}\mathfrak{m}^j\Omega_k(R)$.

(c) If $R = k[[x_1,\ldots,x_n]]$ is the formal power series ring, show that the completion of the module of differentials $\widehat{\Omega_k(R)}$ is the free R-module generated by $dx_1,\ldots,dx_n$.

(d) Let $(R,\mathfrak{m})$ be a complete local ring with coefficient field k, where $\mathfrak{m}$ is generated by elements $y_1,\ldots,y_n$. Show that if we write $R = k[[x_1,\ldots,x_n]]/(f_1,\ldots f_t)$, then

$$\widehat{\Omega_k(R)} = (\oplus R dy_i)/R df_1 + \ldots + R df_t$$

exactly as in the case of an algebra essentially of finite type.

(16) let $R = k[[x_1,\ldots,x_n]]$ be the formal power series ring over a field k with $\text{char}(k) = 0$. Show that if $f \in \mathfrak{m}$, then

$$f \in \overline{\mathfrak{m}(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})}$$

What if R is a polynomial ring?

(17) Let $R = k[[x_1,\ldots,x_n]]$ be the formal power series ring over a field k. Let $\mathfrak{p}$ be a prime ideal in R, and let $f \in \mathfrak{p}^{(m)}$. Show that for all i, $\frac{\partial f}{\partial x_i} \in \mathfrak{p}^{(m-1)}$.

(18) let $R = k[[x_1,\ldots,x_n]]$ be the formal power series ring over a field k with $\text{char}(k) = 0$. Assume that for each $f \in \mathfrak{m} = (x_1,\ldots,x_n)$, f is in $\mathfrak{m}$ times the integral closure of the Jacobian ideal $(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n})$. Prove that for every prime ideal $\mathfrak{p}$ in R,

$$\mathfrak{p}^{(2)} \subset \mathfrak{m}\mathfrak{p}$$

(19) Let k be a field, t a variable over k and $R = k[t^5,t^7,t^{11}]$. Prove that the integral closure of R is $k[t]$ and find the conductor of R.

(20) Let R be a one-dimensional Noetherian domain. Prove that every ideal of $\overline{R}$ that is contained in R is integrally closed.

(21) Let $(R, \mathfrak{m}, k)$ be a complete Noetherian k-algebra and let A be a Noether normalization of R. Assume that R is a complete intersection. Prove that

$$J(R/K) = \mathfrak{d}_K(R/A) = \mathfrak{d}_N(R/A)$$

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