ALGEBRAS WITH STRAIGHTENING LAW

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Contents

1. The Plücker algebra

Let κ be a field, let $1 \leq m \leq n$ be positive integers, and consider the matrix of indeterminates

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

We write $S = \kappa[x_{ij}]$ and consider the subring $R \subset S$ generated by the $m \times m$ (i.e. maximal size) minors of the matrix X. We will see in the course of this week that R is a normal, Cohen-Macaulay graded algebra of dimension $1 + m \cdot (n - m)$.

The ring R is called the **Plücker algebra** and it arises naturally from two closely related constructions:

- Algebraically, we think of S as the ring of polynomial functions on the vector space $\kappa^{m\times n}$ of $m\times n$ matrices. The group $\mathrm{SL}_m(\kappa)$ of $m\times m$ invertible matrices M with $\det(M)=1$ acts on $\kappa^{m\times n}$ via left multiplication, and $R\subset S$ is precisely the subring of $\mathrm{SL}_m(\kappa)$-invariant polynomial functions:

$$R=\{f\in S : f(g\cdot M)=f(M) \text{ for all } g\in \mathrm{SL}_m(\kappa) \text{ and all } M\in\kappa^{m\times n}\}.$$

- Geometrically, R is the homogeneous coordinate ring of the **Grassmannian variety** parametrizing m-dimensional vector subspaces of κ^n.

The Plücker algebra has a nice presentation by generators and relations as follows. For any m-tuple of (not necessarily distinct) columns of X, indexed by $1\leq c_1,\cdots,c_m\leq n$, we consider the $m\times m$ minor

$$[c_1,\cdots,c_m]=\det(x_{i,c_j})_{1\leq i,j\leq m}$$

and note that $[c_1,\cdots,c_m]=0$ unless the c_i are distinct, and that moreover any permutation σ of $c_1,\cdots,c_m$ acts on the minor $[c_1,\cdots,c_m]$ via multiplication by the sign of σ. It is clear that the expressions $[c_1,\cdots,c_m]$ with $1\leq c_1<\cdots<c_m\leq n$ generate R as a κ-algebra. It is then interesting to study the polynomial relations between these generators, which are called the **Plücker relations** and are given in Lemma 1.3 below. Before stating them, we need some preliminary facts.

Let A be any ring, and let M,N be A-modules. A **multilinear map** $f:M^{\oplus q}\longrightarrow N$ is a function with the property that for every $i=1,\cdots,q$ and every fixed choice of $m_1,\cdots,m_{i-1},m_{i+1},\cdots,m_q\in M$ the induced map

$$f(m_1,\cdots,m_{i-1},\bullet,m_{i+1},\cdots,m_q):M\longrightarrow N \text{ is } A\text{-linear.}$$

A multilinear map $f:M^{\oplus q}\longrightarrow N$ is called **alternating** if for every $m_1,\cdots,m_q\in M$ for which there exist $i\neq j$ with $m_i=m_j$ we have

$$f(m_1,\cdots,m_q)=0. \tag{1.1}$$

It follows that for every permutation $\sigma\in\mathfrak{S}_q$ we have

$$f(m_{\sigma(1)},\cdots,m_{\sigma(q)})=\mathrm{sgn}(\sigma)\cdot f(m_1,\cdots,m_q)$$

and this condition is equivalent to the fact that f is alternating when $2=1+1$ is invertible in A.

The main example of an alternating map is the determinant. Let $M=A^{\oplus q}$ and think of the elements of M as column vectors. If we write $[m_1,\cdots,m_q]$ for the determinant of the $q\times q$ matrix whose columns are $m_1,\cdots,m_q$, then the function $f(m_1,\cdots,m_q)=[m_1,\cdots,m_q]$ is alternating. We have

Lemma 1.1. *If $M=A^{\oplus p}$ is a free module of rank p and if $p<q$ then any alternating map $f:M^q\longrightarrow N$ is zero.*

Proof. Fix a basis $\mathbf{e}_1, \cdots, \mathbf{e}_p$ of M and note that by multilinearity the function f is determined by its values on tuples $(\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_q})$ of basis vectors. Any such tuple necessary has repetitions since $q > p$, so f vanishes identically by (1.1). □

Corollary 1.2. *Let A be any ring, fix $1 \leq k \leq m-1$ and consider*

$$v_1, \cdots, v_k, w_{k+2}, \cdots, w_m, u_1, \cdots, u_{m+1} \in A^{\oplus m}.$$

We have

$$\sum_{\sigma \in \mathfrak{S}_{m+1}} \operatorname{sgn}(\sigma) \cdot [v_1, \cdots, v_k, u_{\sigma(1)}, \cdots, u_{\sigma(m-k)}] \cdot [u_{\sigma(m-k+1)}, \cdots, u_{\sigma(m+1)}, w_{k+2}, \cdots, w_m] = 0$$

Proof. Fix $v_1, \cdots, v_k, w_{k+2}, \cdots, w_m$ and consider the left hand side of the above expression as a function $f : M^{\oplus(m+1)} \longrightarrow A$, where $M = A^{\oplus m}$. The function f is alternating and $m < m+1$ hence $f = 0$ by Lemma 1.1. □

Lemma 1.3. *Fix $k \in \{1, \cdots, m-1\}$ consider elements $c_1, \cdots, c_k, d_{k+2}, \cdots, d_m, a_1, \cdots, a_{m+1} \in \{1, \cdots, n\}$. We have*

$$\sum_{\substack{i_1 < \cdots < i_{m-k} \\ i_{m-k+1} < \cdots < i_{m+1} \\ \{i_1, \cdots, i_{m+1}\} = \{1, \cdots, m+1\}}} \operatorname{sgn}(i_\bullet) \cdot [c_1, \cdots, c_k, a_{i_1}, \cdots, a_{i_{m-k}}] \cdot [a_{i_{m-k+1}}, \cdots, a_{i_{m+1}}, d_{k+2}, \cdots, d_m] = 0, \quad (1.2)$$

where we regard $i_\bullet$ as the permutation $\sigma \in \mathfrak{S}_{m+1}$ given by $\sigma(j) = i_j$, and we define $\operatorname{sgn}(i_\bullet)$ to be $\operatorname{sgn}(\sigma)$.

Proof. It is enough to prove this relation when working over $\mathbb{Z}$ (take $\kappa = \mathbb{Z}$). In this case, after multiplying (1.2) by $(m-k)! \cdot (k+1)!$ and using the fact that determinants are alternating functions of the columns of a matrix, the relation (1.2) becomes equivalent to

$$\sum_{\sigma \in \mathfrak{S}_{m+1}} \operatorname{sgn}(\sigma) \cdot [c_1, \cdots, c_k, a_{\sigma(1)}, \cdots, a_{\sigma(m-k)}] \cdot [a_{\sigma(m-k+1)}, \cdots, a_{\sigma(m+1)}, d_{k+2}, \cdots, d_m] = 0$$

This identity is a special case of Corollary 1.2, which concludes our proof. □

Example 1.4. If $m = 2$, $n = 4$ and $k = 1$ then we get

$$[1, 2] \cdot [3, 4] - [1, 3] \cdot [2, 4] + [1, 4] \cdot [2, 3] = 0.$$

We will represent a product of d maximal minors as a $d \times m$ array which we will call a **tableau**, where a row with entries $c_1, \cdots, c_m$ corresponds to the minor $[c_1, \cdots, c_m]$. For instance the equalities in

Example 1.4 will be written as (we color blue the boxes corresponding to the a_i's in the Plücker relation)

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} \tag{1.3}$$

We will call a tableau **standard** if it is strictly increasing along rows and weakly increasing along columns (in the literature these tableaux, or their transposed versions, are typically called **semi-standard**!). For instance (we color red the boxes that exhibit the failure of standardness)

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 2 & 3 \\ \hline \end{array} \text{ is standard, but } \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \text{ are not.}$$

The Plücker relations can be pictured in terms of (two row) tableaux as

$$\sum_{\substack{i_1<\cdots<i_{m-k} \\ i_{m-k+1}<\cdots<i_{m+1} \\ \{i_1,\cdots,i_{m+1}\}=\{1,\cdots,m+1\}}} \operatorname{sgn}(i_\bullet)\cdot \begin{array}{|c|c|c|c|c|c|c|}\hline c_1 & \cdots & c_k & a_{i_1} & \cdots & \cdots & a_{i_{m-k}} \\ \hline a_{i_{m-k+1}} & \cdots & \cdots & a_{i_{m+1}} & d_{k+2} & \cdots & d_m \\ \hline \end{array} = 0 \tag{1.4}$$

They induce relations between $d\times m$ tableaux for any $d\geq 2$ in the obvious way. For instance multiplying the relations (1.3) by $[2,4]\cdot[2,3]$ we obtain (we color green the rows that are not involved in the Plücker relation)

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 2 & 4 \\ \hline 2 & 3 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 2 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline 2 & 4 \\ \hline 2 & 3 \\ \hline \end{array} = 0$$

where all the interesting action takes place in the first two rows.

In what follows we will consider the grading of the Plücker algebra obtained by declaring that $[c_1,\cdots,c_m]$ has degree one (note that this differs from the grading induced from the standard grading on S by a factor of $1/m$). We will also think of tableaux as elements of R (or S) as explained earlier. With this convention we have

Theorem 1.5. *The degree d component R_d of the Plücker algebra has a basis consisting of standard $d\times m$ tableaux with entries in $\{1,\cdots,n\}$. In particular, any $d\times m$ tableau can be expressed uniquely as a linear combination of standard tableaux.*

The linear independence of standard tableaux follows from the following easy linear algebra fact:

Lemma 1.6. *Let E be a vector space with a totally ordered basis $\mathbf{e}_1, \mathbf{e}_2, \cdots$, and for every vector*

$$0 \neq \mathbf{v} = \sum_i a_i \cdot \mathbf{e}_i \text{ in } E$$

define the leading term of $\mathbf{v}$ to be $\mathrm{lt}(\mathbf{v}) = \mathbf{e}_{i_0}$ if $a_{i_0} \neq 0$ and $a_i = 0$ for $i < i_0$. If $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are non-zero vectors in E with distinct leading terms then they are linearly independent.

If we order the pairs $\{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ **lexicographically**, i.e.

$$(1,1) < (1,2) < \cdots < (1,n) < (2,1) < (2,2) < \cdots < (m,n) \tag{1.5}$$

then this induces a **lexicographic ordering** of the monomials in S given by

$$\prod x_{ij}^{a_{ij}} < \prod x_{ij}^{b_{ij}} \text{ if for the smallest } (i,j) \text{ for which } a_{ij} \neq b_{ij} \text{ we have } a_{ij} > b_{ij}.$$

This yields a total ordering of the monomial basis of the κ-vector space S as in Lemma 1.6, with the property that

$$\mathrm{lt}([c_1, \cdots, c_m]) = x_{1,c_1} \cdot x_{2,c_2} \cdots x_{m,c_m} \text{ if } c_1 < c_2 < \cdots < c_m.$$

In particular if T, T' are tableaux with increasing rows then $\mathrm{lt}(T) = \mathrm{lt}(T')$ if and only if for each $k = 1, \cdots, m$ the entries in the k-th column of T and T' are the same (up to permutation). Since no non-trivial permutation of a weakly increasing sequence is weakly increasing, it follows that distinct standard tableaux have distinct leading terms and Lemma 1.6 applies to prove that standard tableaux are linearly independent. To prove that standard tableaux span R we show the following stronger result.

Lemma 1.7. *Every $d \times m$ tableau T can be expressed modulo the Plücker relations as a linear combination of standard tableaux.*

Proof. We say that a tableau T is **normalized** if each of its rows is strictly increasing, and if the rows are arranged in increasing lexicographic order. Any tableau T either has repeated entries in some row (in which case $T = 0$ in R) or it is equal up to sign with a normalized tableau (permuting the entries in a row of a tableau can only affect the sign, while permuting the rows has no effect on the corresponding polynomial; we can thus first sort the entries in each row, and then sort the rows lexicographically in order to obtain a normalized tableau). It then suffices to show that every normalized tableau can be expressed modulo the Plücker relations as a linear combination of standard tableaux.

If we identify a $d \times m$ tableau with a string of length $d \cdot m$ by reading its entries from left to right and top to bottom, then we can order the normalized tableaux by considering the lexicographic ordering of the corresponding strings. Assume that there exists a normalized tableau which cannot be expressed modulo the Plücker relations as a linear combination of standard tableaux, and consider T to be the lexicographically smallest such tableau. Let $(i, k+1)$ with $1 \leq i \leq d$ and $1 \leq k+1 \leq m$ be the lexicographically smallest pair such that if we let $[c_1, \cdots, c_m]$ be the i-th row of T, and $[d_1, \cdots, d_m]$ be the $(i+1)$-st row, then $c_{k+1} > d_{k+1}$. We define $a_1, \cdots, a_{m+1}$ by the equality

$$(a_1, \cdots, a_{m+1}) = (c_{k+1}, \cdots, c_m, d_1, \cdots, d_{k+1})$$

and apply Lemma 1.3. Using the Plücker relation (1.4) and renormalizing the resulting tableaux allows us to express T as a linear combination of lexicographically smaller normalized tableaux. By the minimality of T, each such tableau is a linear combination of standard tableaux modulo the Plücker relations, so the same must be true about T. This contradicts the assumption on T and concludes the proof. □

Theorem 1.5 and Lemma 1.7 show that the Plücker relations generate all the polynomial relations between the $m \times m$ minors of X. To state this precisely, we let $[n] = \{1, \cdots, n\}$ and write

$$\binom{[n]}{m} = \{(c_1, \cdots, c_m) : 1 \leq c_1 < c_2 < \cdots < c_m \leq n\}$$

for the collection of m-element subsets of $[n]$. It will be important to consider the partial order on $\binom{[n]}{m}$ given by

$$(c_1, \cdots, c_m) \leq (d_1, \cdots, d_m) \text{ if and only if } c_i \leq d_i \text{ for all } i = 1, \cdots, m. \tag{1.6}$$

We say that $\binom{[n]}{m}$ is a **poset** (partially ordered set) and we call it the **Plücker poset**. We consider the polynomial ring

$$P = \kappa\left[p_{\underline{c}} : \underline{c} \in \binom{[n]}{m}\right]$$

and the surjective ring homomorphism $\pi : P \twoheadrightarrow R$ defined by $p_{\underline{c}} \mapsto [c_1, \cdots, c_m]$.

Theorem 1.8. *The kernel of π is the ideal generated by the quadratic Plücker relations determined by Lemma 1.3.*

Proof. Let I be the ideal generated by the Plücker relations and note that $I \subseteq \ker(\pi)$. To show the equality, consider any polynomial $f \in \ker(\pi)$: using the Plücker relations as in Lemma 1.7 we can express

$$f = g + \sum_i a_i \cdot (p_{\underline{c}^1} \cdot p_{\underline{c}^2} \cdots)$$

where $a_i \in \kappa$, $g \in I$ and $\underline{c}^1 \leq \underline{c}^2 \leq \cdots$. Applying the map π and using the fact that $\pi(f) = 0$ (by assumption) and that $\pi(g) = 0$ (since $I \subseteq \ker(\pi)$) it follows that

$$0 = \sum_i a_i \cdot ([\underline{c}^1] \cdot [\underline{c}^2] \cdots)$$

Since $\underline{c}^1 \leq \underline{c}^2 \leq \cdots$ it follows that each product $[\underline{c}^1] \cdot [\underline{c}^2] \cdots$ is equal to a standard tableau. Since standard tableaux are linearly independent we obtain that all coefficients $a_i = 0$ and hence $f = g \in I$, concluding the proof. □

The polynomial ring P is the homogeneous coordinate ring of the projective space $\mathbb{P}^{\binom{n}{m}-1}$. The ideal $I = \ker(\pi)$ of Plücker relations defines a projective algebraic variety called the **Grassmann variety**. The points of this variety are in bijective correspondence with the collection of m-dimensional subspaces of κ^n. The presentation of the Plücker algebra as a quotient of a polynomial ring is just the first step in the construction of its minimal free resolution. It is then natural to ask the following

Open Question 1.9. What is the minimal free resolution of R as a P-module?

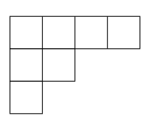
In the case when $m = 2$ or $m = n - 2$, the Plücker algebra can be identified with the coordinate ring of the algebraic variety of skew-symmetric matrices of rank ≤ 2. When $\text{char}(\kappa) = 0$, the minimal free resolution (as well as the corresponding resolutions of coordinate rings of matrices of higher rank) can be found in [Wey03, Section 6.4].

A natural variant of the Plücker algebra is obtained by considering lower order minors. For $1 \leq t \leq m$ define $R(t)$ to be the κ-algebra generated by the $t \times t$ minors of the matrix X. In this case we do not even know the presentation of the algebra:

Open Question 1.10. What are the defining relations of the algebra $R(t)$ for $1 < t < m$?

This problem has been investigated in [BCV13] in the case when $\text{char}(\kappa) = 0$, where the authors conjectured that the ideal of relations is generated by quadratic and cubic polynomials, and gave a complete conjectural description of these generators.

It will be useful for the exercises below, as well as in the last section where we discuss determinantal varieties, to consider more general tableaux than the rectangular ones. First of all recall that a **partition** λ is a non-increasing sequence of natural numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, and can be represented pictorially by a **Young diagram** of left justified rows of boxes, having λ_i boxes in row i. For example, $\lambda = (4, 2, 1)$ corresponds to the Young diagram



We identify any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with a subset $\lambda \subset \mathbb{N} \times \mathbb{N}$ consisting of all the pairs (i, j) with $1 \leq j \leq \lambda_i$, so $\lambda = (4, 2, 1)$ correponds to

$$\lambda = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1)\}.$$

Given a partition $\lambda \subset \mathbb{N}\times\mathbb{N}$, a **tableau T of shape λ** is a function $T : \lambda \longrightarrow \mathbb{N}$, or equivalently a filling of the boxes of the Young diagram of λ with natural numbers. We write $|T|$ for the Young diagram defining the shape of T. A tableau T is **standard** if it is strictly increasing along rows and weakly increasing down columns. When $\lambda = (m, m, \cdots, m)$ has d parts equal to m, the Young diagram of λ is just a $d \times m$ rectangle, and a tableau of shape λ is just a $d \times m$ rectangular array of numbers as discussed earlier. You will explore the following question in the exercises:

Question 1.11. What is the number $k_\lambda(n)$ of standard tableaux of shape λ with entries in $\{1, 2, \cdots, n\}$?

The numbers $k_\lambda(n)$ are called **Kostka numbers** and they compute dimensions of irreducible representations of the group GL_n. A general formula (which we won't prove here) is given as follows. Consider the tableau T defined by

$$T(i, j) = n + i - j \text{ for } (i, j) \in \lambda.$$

When $\lambda = (4, 2, 1)$ and $n = 5$ we get

$$T = \begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 2 \\ \hline 6 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Given a partition λ, the **hook** $H_\lambda^{i,j}$ centered at (i, j) is the subset

$$H_\lambda^{i,j} = \{(a, b) \in \lambda : a = i \text{ and } b \geq j, \text{ or } a \geq i \text{ and } b = j\}.$$

The length of the hook $H_\lambda^{i,j}$ is $h_\lambda^{i,j} = |H_\lambda^{i,j}|$. We then consider the tableaux that records the hook lengths inside the Young diagram of λ:

$$H(i, j) = h_\lambda^{i,j} \text{ for } (i, j) \in \lambda.$$

When $\lambda = (4, 2, 1)$ and $n = 5$ we get

$$H = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Whit this notation, the Kostka number $k_\lambda(n)$ can be computed via:

$$k_\lambda(n) = \prod_{(i,j)\in\lambda} \frac{T(i, j)}{H(i, j)}. \tag{1.7}$$

When $\lambda = (4, 2, 1)$ and $n = 5$ we get $k_{4,2,1}(5) = 175$.

1.1. Exercises.

(1) Show that the collection of all the minors of the generic matrix X (of any size from 1 up to m) can be realized as Plücker coordinates of the **extended matrix**

$$X^e = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 0 & \cdots & 0 & 1 \\ x_{21} & x_{22} & \cdots & x_{2n} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & 1 & \cdots & 0 & 0 \end{bmatrix}$$

(2) Show that the Plücker algebra associated to the generic $m \times n$ matrix is isomorphic to that associated to the generic $(n-m) \times n$ matrix.

(3) Consider the generic $n \times n$ skew-symmetric matrix

$$X^{skew} = \begin{bmatrix} 0 & p_{12} & p_{13} & \cdots & p_{1n} \\ -p_{12} & 0 & p_{23} & \cdots & p_{2n} \\ -p_{13} & -p_{23} & 0 & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{1n} & -p_{2n} & -p_{3n} & \cdots & 0 \end{bmatrix}$$

Show that the ideal of relations defining the Plücker algebra for $m = 2$ is the same as the ideal generated by the Pfaffians of the principal 4×4 submatrices of X^{skew}.

(4) The purpose of this exercise is to verify (1.7) in a few special cases.

(a) Show that if $\lambda_1 > n$ then there are no standard tableaux of shape λ with entries in $\{1, \cdots, n\}$. Verify that this agrees with (1.7).

(b) Let $h_a(n) = \binom{a+n-1}{a}$. Prove that for the partition $\lambda = (1^a) = (1, \cdots, 1)$ we have

$$k_{1^a}(n) = h_a(n)$$

and check that this agrees with (1.7).

(c) Let $e_a(n) = \binom{n}{a}$. Prove that for the partition $\lambda = (a)$ we have

$$k_a(n) = e_a(n)$$

and check that this agrees with (1.7).

(d) Prove by induction on n that for $a \geq b$ we have $k_{a,b}(n) = e_a(n) \cdot e_b(n) - e_{a+1}(n) \cdot e_{b-1}(n)$. Use this to prove (1.7) when $\lambda = (a, b)$.

(e) Prove by induction on n that for $a \geq b$ we have $k_{2^b,1^{a-b}}(n) = h_a(n) \cdot h_b(n) - h_{a+1}(n) \cdot h_{b-1}(n)$. Use this to prove (1.7) when $\lambda = (2^{a-b}, 1^b)$.

(5) Show that in the case when $m = 2$, the Hilbert function $H_R(d)$ of the Plücker algebra is given by

$$H_R(d) = \frac{1}{d+1} \cdot \binom{n+d-1}{d} \cdot \binom{n+d-2}{d}.$$

Verify that in this case $\dim(R) = 2n - 3$ and that the **multiplicity/degree** of R (with respect to the maximal homogeneous ideal) is given by the **Catalan number**

$$e(R) = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

(6) Use formula (1.7) to derive the dimension and the degree of R for arbitrary m, n.

(7) Determine the number of minimal generators of the ideal of Plücker relations for arbitrary m, n.

(8) Prove that for every $1 \leq k \leq m$ one has

$$[c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m] = \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m} [c_1, \cdots, c_k, d_{i_1}, \cdots, d_{i_{m-k}}] \cdot [d_1, \cdots, c_{k+1}, \cdots, c_m, \cdots, d_m]$$

where $[d_1, \cdots, c_{k+1}, \cdots, c_m, \cdots, d_m]$ denotes the tuple obtained from $[d_1, \cdots, d_m]$ by replacing the elements $d_{i_1}, \cdots, d_{i_{m-k}}$ with $c_{k+1}, \cdots, c_m$ (in this order). Pictorially, one has

$$\begin{array}{|c|c|c|}\hline c_1 & \cdots & c_m \\ \hline d_1 & \cdots & d_m \\ \hline \end{array} = \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline c_1 & c_2 & \cdots & \cdots & \cdots & c_k & d_{i_1} & \cdots & \cdots & \cdots & d_{i_{m-k}} \\ \hline d_1 & \cdots & \cdots & c_{k+1} & \cdots & \cdots & \cdots & \cdots & c_m & \cdots & d_m \\ \hline \end{array}$$

Use these relations to give an alternative proof of the fact that any tableau is a linear combination of standard tableaux.

2. Algebras with straightening law

Suppose that A is a ring and that $H \subset A$ is a finite subset which is partially ordered (H is a **poset**). A **standard monomial** is a product of a totally ordered set of elements of H:

$$\alpha_1 \cdots \alpha_k \text{ where } \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k.$$

Assume now that A is a K-algebra for some ring K and that the elements of H generate A as a K-algebra. We say that A is an **algebra with straightening law** (on H, over K) if it satisfies the following axioms:

(ASL-1) The algebra A is a free K-module with basis given by the set of standard monomials.

(ASL-2) If $\alpha, \beta \in H$ are incomparable and if

$$\alpha \cdot \beta = \sum_i c_i \cdot (\gamma_{i1} \cdot \gamma_{i2} \cdots) \tag{2.1}$$

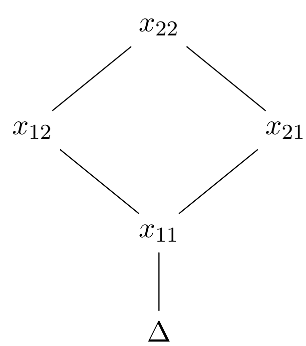
is the unique expression of $\alpha \cdot \beta$ as a linear combination of standard monomials (here $c_i \neq 0$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots$) then $\gamma_{i1} < \alpha, \beta$ for all i.

The relations in (ASL-2) are called the **straightening relations** of the algebra A.

Example 2.1 (Polynomial rings)**.** If H is a totally ordered set (for short a **chain**) then A is necessarily a polynomial ring $K[H]$ on the elements of H.

Example 2.2 (The discrete ASL)**.** If $\alpha \cdot \beta = 0$ for every pair of incomparable elements $\alpha, \beta \in H$ then A is called the **discrete ASL on** H. As a consequence of Theorem 2.9 below, a discrete ASL is isomorphic to the quotient of a polynomial ring by an ideal generated by square-free monomials of degree two. In Exercise 1 you will show that in fact every quotient of a polynomial ring by a square-free monomial ideals has the structure of an ASL.

Example 2.3 (2×2 matrices)**.** Consider the polynomial ring $S = \kappa[x_{11}, x_{12}, x_{21}, x_{22}]$, let $\Delta = x_{11} \cdot x_{22} - x_{12} \cdot x_{21}$ denote the determinant of the generic 2×2 matrix, let $H = \{x_{11}, , x_{12}, x_{21}, x_{22}, \Delta\}$ and consider the partial order on H defined by the following Hasse diagram (with the convention that smaller elements are at the bottom)



The only incomparable pair of elements is x_{12}, x_{21} and the straightening relation is

$$x_{12} \cdot x_{21} = x_{11} \cdot x_{22} - \Delta.$$

Example 2.4 (The Plücker algebra)**.** Let $1 \leq m \leq n$, let R be the Plücker algebra defined in Section 1 and consider $H = \{[c_1, \cdots, c_m] : \underline{c} \in \binom{[n]}{m}\} \subset R$ with the partial order induced by (1.6). Because of the identification between H and $\binom{[n]}{m}$, we will also refer to H as the **Plücker poset**. The standard monomials correspond precisely to the standard tableaux in Section 1, while the straightening relations in (ASL-2) come from the Plücker relations as follows. You first need to check:

Lemma 2.5. *Suppose that* $[c_1, \cdots, c_m]$, $[d_1, \cdots, d_m]$ *are elements of the Plücker poset, and that for some* $k \in \{0, \cdots, m-1\}$ *we have* $c_i \le d_i$ *for all* $i = 1, \cdots, k$, *and* $c_{k+1} > d_{k+1}$. *Show that (1.2) allows one to write*

$$[c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m] = \sum \pm [e_1, \cdots, e_m] \cdot [f_1, \cdots, f_m]$$

where each term satisfies

$$[e_1, \cdots, e_m] \le [c_1, \cdots, c_m] \text{ and } e_i \le f_i \text{ for } i = 1, \cdots, k+1.$$

Iterating this we obtain

$$[c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m] = \sum_{c_{\mathbf{e},\mathbf{f}}} c_{\mathbf{e},\mathbf{f}} \cdot [e_1, \cdots, e_m] \cdot [f_1, \cdots, f_m] \tag{2.2}$$

where $c_{\mathbf{e},\mathbf{f}} \in \kappa$, $[e_1, \cdots, e_m] \le [f_1, \cdots, f_m]$ *and* $[e_1, \cdots, e_m] \le [c_1, \cdots, c_m]$

Corollary 2.6. *The Plücker algebra* R *is an ASL on the Plücker poset.*

Proof. We already know by Theorem 1.5 that the standard monomials form a basis of R, so condition (ASL-1) is satisfied. To check (ASL-2) we apply Lemma 2.5: observe that in the relation (2.2) all the monomials $[e_1, \cdots, e_m] \cdot [f_1, \cdots, f_m]$ are standard, so (2.2) is the unique representation of $[c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m]$ as a linear combination of standard monomials.

We can now reverse the roles of $[c_1, \cdots, c_m]$ and $[d_1, \cdots, d_m]$ and apply Lemma 2.5 to express

$$[d_1, \cdots, d_m] \cdot [c_1, \cdots, c_m] = \sum_{c'_{\mathbf{e},\mathbf{f}} \neq 0} c'_{\mathbf{e},\mathbf{f}} \cdot [e_1, \cdots, e_m] \cdot [f_1, \cdots, f_m]$$

where $c_{\mathbf{e},\mathbf{f}} \in \kappa$, $[e_1, \cdots, e_m] \le [f_1, \cdots, f_m]$ and $[e_1, \cdots, e_m] \le [d_1, \cdots, d_m]$. Since $[d_1, \cdots, d_m] \cdot [c_1, \cdots, c_m] = [c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m]$ we conclude that the above expression is identical to (2.2), in particular $c_{\mathbf{e},\mathbf{f}} = c'_{\mathbf{e},\mathbf{f}}$ and most importantly

$$[e_1, \cdots, e_m] \le [c_1, \cdots, c_m], [d_1, \cdots, d_m] \text{ for all terms with } c_{\mathbf{e},\mathbf{f}} \neq 0$$

This shows that condition (ASL-2) is satisfied for the Plücker algebra, which concludes our proof. □

2.1. **Graded families of ideals and associated graded rings.** Consider a descending family of ideals $I_\bullet$ in a ring A:

$$I_\bullet : \quad A = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

We say that $I_\bullet$ is a **multiplicative family of ideals** if

$$I_p \cdot I_q \subseteq I_{p+q} \text{ for all } p, q \ge 0.$$

We will assume throughout that

$$\bigcap_{j\geq 0} I_j = 0. \tag{2.3}$$

Given a multiplicative family of ideals $I_\bullet$ we consider the **associated graded ring**

$$\mathrm{gr}_{I_\bullet}(A) = \mathrm{gr}(A) = \bigoplus_{j\geq 0} I_j/I_{j+1} = A/I_1 \oplus I_1/I_2 \oplus \cdots$$

where the multiplication is induced by that of A. Given an element $a \in A$ we define the **initial term** $\mathrm{in}(a) \in \mathrm{gr}(A)$ as follows:

- $\mathrm{in}(a) = 0$ if $a \in I_j$ for all j, which by virtue of (2.3) is equivalent to $a = 0$; or else
- consider the maximum j such that $a \in I_j$ and let $\mathrm{in}(a) = \overline{a} \in I_j/I_{j+1}$.

The association $a \mapsto \mathrm{in}(a)$ is in general not a ring homomorphism, but it satisfies the property that for every $a, b \in A$ we either have that $\mathrm{in}(a)\cdot\mathrm{in}(b) = \mathrm{in}(ab)$ or $\mathrm{in}(a)\cdot\mathrm{in}(b) = 0$. In Exercise 4 you will show that if $\mathrm{gr}(A)$ is a domain (resp. reduced) then the ring A is also a domain (resp. reduced). In general the philosophy is that $\mathrm{gr}(A)$ is more poorly behaved than A so whenever we can say that $\mathrm{gr}(A)$ has nice properties then they must be already present in A.

An alternative construction of $\mathrm{gr}(A)$ proceeds as follows. Consider an auxiliary variable t and define the **Rees algebra** associated to $I_\bullet$ as a subalgebra of the Laurent polynomial ring $A[t, t^{-1}]$ via

$$\mathcal{R}(I_\bullet) = \mathcal{R} = \cdots \oplus At^k \oplus \cdots \oplus At \oplus A \oplus I_1 t^{-1} \oplus \cdots \oplus I_k t^{-k} \oplus \cdots$$

In Exercise 5 you will verify that $\mathrm{gr}(A)$ is isomorphic to the quotient $\mathcal{R}/(t)$.

Assume now that A is an ASL on a poset H over a ring K and consider a multiplicative family of ideals $I_\bullet$. We define the **order** of a standard monomial with respect to the family $I_\bullet$ as follows.

- If $\alpha \in H$ then we let
$$\mathrm{ord}_{I_\bullet}(\alpha) = \max\{j : \alpha \in I_j\}.$$
- If $\alpha_1 \cdots \alpha_k$ is a standard monomial then we let
$$\mathrm{ord}_{I_\bullet}(\alpha_1\cdots\alpha_k) = \sum_{i=1}^{k} \mathrm{ord}_{I_\bullet}(\alpha_i).$$

We say that the filtration $I_\bullet$ is **standard** if for each $j \geq 0$ the ideal I_j is spanned (over K) by the standard monomials of order $\geq j$.

Theorem 2.7. *If A is an ASL on H over K and if $I_\bullet$ is a standard multiplicative family of ideals then the associated Rees algebra $\mathcal{R}$ is an ASL on H over $K[t]$, where we think of H as a subset in $\mathcal{R}$ via the inclusion given by*

$$\alpha \mapsto \tilde{\alpha} = t^{-\operatorname{ord}(\alpha)} \cdot \alpha.$$

Moreover, the associated graded ring $\operatorname{gr}(A)$ *is an ASL on H over K, where we think of H as a subset in* $\operatorname{gr}(A)$ *via the inclusion given by*

$$\alpha \mapsto \overline{\alpha} = \operatorname{in}(\alpha).$$

Proof. We will prove that $\mathcal{R}$ is an ASL on H over $K[t]$, which implies by base change (see Exercise 6) that

$$\operatorname{gr}(A) \overset{\text{Exercise 5}}{=} \mathcal{R}/t\mathcal{R} = \mathcal{R} \otimes_{K[t]} K[t]/tK[t]$$

is an ASL on H over $K[t]/tK[t] = K$. The natural quotient map $\mathcal{R} \longrightarrow \mathcal{R}/t\mathcal{R}$ sends $\tilde{\alpha} \longrightarrow \overline{\alpha}$, so the poset defining the ASL structure on $\operatorname{gr}(A)$ will consist of $\{\overline{\alpha} : \alpha \in H\}$, as desired.

To check the ASL axioms for $\mathcal{R}$ we begin with (ASL-1). We first note that since A is a free K-module with basis consisting of the standard monomials it follows that $A[t, t^{-1}]$ is a free $K[t, t^{-1}]$-module on the same basis. Rescaling each basis elements by a unit (for instance by a negative power of t) will also result in a $K[t, t^{-1}]$-basis of $A[t, t^{-1}]$, hence

$$\tilde{\alpha_1} \cdots \tilde{\alpha_k} \text{ where } \alpha_1 \leq \cdots \leq \alpha_k,\ \alpha_i \in H, \tag{2.4}$$

form a $K[t, t^{-1}]$-basis of $A[t, t^{-1}]$. Since they are linearly independent over $K[t, t^{-1}]$ it follows that they must also be linearly independent over $K[t]$. To finish the proof of (ASL-1) we have to check that the standard monomials (2.4) span $\mathcal{R}$ as a $K[t]$-module.

Consider first $f \in At^k$ for $k \geq 0$ and write $f = a \cdot t^k$ with $a \in A$. Since a is a K-linear combination of standard monomials in A we get

$$a = \sum c_i \cdot (\alpha_1 \cdot \alpha_2 \cdots), \text{ with } \alpha_1 \leq \alpha_2 \leq \cdots$$

This means that

$$f = a \cdot t^k = \sum c_i \cdot t^{k+\operatorname{ord}(\alpha_1)+\operatorname{ord}(\alpha_2)+\cdots} \cdot (\tilde{\alpha}_1 \cdot \tilde{\alpha}_2 \cdots)$$

is a $K[t]$-linear combination of the standard monomials (2.4).

Consider now $f \in I_k t^{-k}$ for $k \geq 0$. Since $I_\bullet$ is a standard family, we can write $f = at^{-k}$ where a is a linear combination of standard monomials of order $>= k$. It follows that in the expression

$$f = a \cdot t^{-k} = \sum c_i \cdot (\tilde{\alpha}_1 \cdot \tilde{\alpha}_2 \cdots) \cdot t^{-k+\operatorname{ord}(\alpha_1)+\operatorname{ord}(\alpha_2)+\cdots}$$

all the exponents $-k+\mathrm{ord}(\alpha_1)+\mathrm{ord}(\alpha_2)+\cdots$ are non-negative, and therefore f is again a $K[t]$-linear combination of the standard monomials (2.4). This concludes the proof of (ASL-1).

You will verify (ASL-2) in Exercise 7. □

Using the observation that A is often **better** than $\mathrm{gr}(A)$, we will apply Theorem 2.7 to deduce good properties of A from similar properties of $\mathrm{gr}(A)$. In fact as we will see shortly it is possible to iterate the construction $A \longrightarrow \mathrm{gr}(A)$ by choosing at each step appropriate multiplicative families of ideals, in such a way that at the very last step we obtain a discrete ASL (Example 2.2). Since discrete ASLs are easier to understand, the hope is that this would shed some light on the algebra A that we started with.

To measure the failure of an ASL of being discrete, we introduce a new invariant. For A an ASL we define its **indiscrete set** $\mathrm{Ind}(A) \subseteq H$ to be the set of all elements $\gamma \in H$ that appear on the right hand side of some straightening relation in (2.1). The **indiscreteness** of A is the size $\mathrm{ind}(A) = |\,\mathrm{Ind}(A)|$ of its indiscrete set. Note that A is a discrete ASL if and only if $\mathrm{ind}(A)=0$.

Theorem 2.8. *Let α be a minimal element of* $\mathrm{Ind}(A)$ *and consider the multiplicative family of ideals $I_\bullet$ with $I_n=(\alpha^n)$. We have that $I_\bullet$ is a standard family, and that (after the identification $\alpha \longleftrightarrow \overline{\alpha}$ for $\alpha \in H$) we have*

$$\mathrm{Ind}(\mathrm{gr}(A)) \subsetneq \mathrm{Ind}(A).$$

Proof. Notice that $I_\bullet$ is a multiplicative family, so for the first assertion we need to check that it is standard. Since A is spanned by standard monomials, $I_n=(\alpha^n)$ is spanned by products $\alpha^n\cdot M$ where M is a standard monomial. Notice that if $\beta\in H$ is such that α,β are incomparable then $\alpha\cdot\beta=0$: otherwise the right hand side of (2.1) would be non-empty so we could find an element $\gamma_{i1}\in\mathrm{Ind}(A)$ with $\gamma_{i1}<\alpha,\beta$, which would contradict the minimality assumption on α. It follows that $\alpha^n\cdot M=0$ if M contains factors β incomparable to α. Since $\alpha^n\cdot M$ is standard whenever all factors β of M are comparable to α, we conclude that I_n is spanned by standard monomials.

We now need to verify that $\mathrm{Ind}(\mathrm{gr}(A)) \subsetneq \mathrm{Ind}(A)$. Since the ASL structure on $\mathrm{gr}(A)$ is induced from the ASL structure on $\mathcal{R}$ by base change, it is clear that $\mathrm{Ind}(\mathrm{gr}(A))\subseteq\mathrm{Ind}(\mathcal{R})$. By Exercise 7 we get that $\mathrm{Ind}(\mathcal{R})\subseteq\mathrm{Ind}(A)$ and therefore

$$\mathrm{Ind}(\mathrm{gr}(A)) \subseteq \mathrm{Ind}(A).$$

To prove that the inclusion is strict, we will show that $\alpha\notin\mathrm{Ind}(\mathrm{gr}(A))$. Suppose otherwise, and consider a straightening relation (we write $\overline{\bullet}$ for the reduction mod t map $\mathcal{R}\longrightarrow\mathcal{R}/t\mathcal{R}=\mathrm{gr}(A)$)

$$\overline{\beta}\cdot\overline{\delta}=\sum_i \overline{c_i\cdot t^{m_i-\mathrm{ord}_{I_\bullet}(\beta)-\mathrm{ord}_{I_\bullet}(\delta)}}\cdot(\overline{\gamma_{i1}}\cdot\overline{\gamma_{i2}}\cdots)$$

where $m_i = \mathrm{ord}_{I_\bullet}(\gamma_{i1}) + \mathrm{ord}_{I_\bullet}(\gamma_{i2}) + \cdots$ and such that $\overline{\alpha}$ appears as a factor in a non-zero term on the right hand side. By the minimality of α we must have $\gamma_{i1} = \alpha$ for some i for which $m_i = \mathrm{ord}_{I_\bullet}(\beta) + \mathrm{ord}_{I_\bullet}(\delta)$. Note that if $\mathrm{ord}_{I_\bullet}(\beta) + \mathrm{ord}_{I_\bullet}(\delta) > 0$ then $\beta = \alpha$ or $\delta = \alpha$, in which case we observed at the beginning of the proof that $\beta \cdot \delta = 0$ since β, δ are incomparable. It follows that $\overline{\beta} \cdot \overline{\delta} = 0$, which contradicts the fact that $\overline{\alpha}$ appears on the right hand side! We may thus assume that $m_i = \mathrm{ord}_{I_\bullet}(\beta) + \mathrm{ord}_{I_\bullet}(\delta) = 0$, but

$$m_i = \mathrm{ord}_{I_\bullet}(\gamma_{i1}) + \mathrm{ord}_{I_\bullet}(\gamma_{i2}) + \cdots \geq \mathrm{ord}_{I_\bullet}(\gamma_{i1}) = \mathrm{ord}_{I_\bullet}(\alpha) = 1,$$

which is again a contradiction. This means that $\alpha \notin \mathrm{Ind}(\mathrm{gr}(A))$ which concludes our proof. □

2.2. **Two basic properties of ASLs.** Recall that an ASL A on H over K is generated as a K-algebra by the elements of H and that moreover these elements satisfy the straightening relations in (ASL-2). But there is no a priori reason why there could be no other relations between the elements of H besides the ones implied by the straightening relations. Nevertheless, we can prove that indeed there aren't any:

Theorem 2.9. *Consider the polynomial ring $P = K[T_\alpha : \alpha \in H]$ and the K-algebra homomorphism $\pi : P \twoheadrightarrow A$ defined by $T_\alpha \mapsto \alpha$. The ideal $\ker(\pi)$ is generated by the polynomials*

$$T_\alpha \cdot T_\beta - \sum_i c_i \cdot (T_{\gamma_{i1}} T_{\gamma_{i2}} \cdots) \tag{2.5}$$

where α, β run over all pairs of incomparable elements of H and where c_i, γ_{ij} are as in (2.1).

Proof. We consider any total ordering of the variables T_α which is compatible with the partial ordering on H, i.e. $T_\alpha < T_\beta$ for all $\alpha, \beta \in H$ with $\alpha < \beta$. This induces a lexicographic ordering of the monomials in $K[T_\alpha]$ given as follows: first of all we say that a monomial $M = T_{\alpha_1} T_{\alpha_2} \cdots$ is **normalized** if $T_{\alpha_1} \leq T_{\alpha_2} \leq \cdots$. For every two normalized monomials

$$M = T_{\alpha_1} T_{\alpha_2} \cdots \text{ and } M' = T_{\beta_1} T_{\beta_2} \cdots \text{ we say}$$

$$M < M' \text{ if and only if for the smallest } i \text{ for which } \alpha_i \neq \beta_i \text{ we have } T_{\alpha_i} < T_{\beta_i}.$$

We say that a monomial $M = T_{\alpha_1} T_{\alpha_2} \cdots$ is **standard** if $\alpha_1 \leq \alpha_2 \leq \cdots$. By (ASL-1) the map $T_\alpha \longrightarrow \alpha$ establishes an isomorphism between the K-span of standard monomials $T_{\alpha_1} T_{\alpha_2} \cdots$ and A, so in order to prove that $\ker(\pi)$ is generated by (2.5) it is enough to check that every normalized monomial can be expressed modulo the relations (2.5) as a linear combination of standard monomials. Suppose that this isn't the case, and consider the lexicographically minimal (normalized) counterexample $M = T_{\alpha_1} T_{\alpha_2} \cdots$.

Since M is not standard, we can find i such that $\alpha_i \not\leq \alpha_{i+1}$. We consider the minimal such index i and observe that $\alpha_{i+1} \not\leq \alpha_i$: otherwise we would get $T_{\alpha_{i+1}} < T_{\alpha_i}$ by the choice of the ordering on T_α's, but this would contradict the fact that M was normalized. Modulo the polynomial

$$T_{\alpha_i} \cdot T_{\alpha_{i+1}} - \sum_j c_j \cdot (T_{\gamma_{j1}} T_{\gamma_{j2}} \cdots), \text{ with } \gamma_{j1} < \alpha_i, \alpha_{i+1},$$

we can rewrite M as a linear combination of

$$T_{\alpha_1} \cdots T_{\alpha_{i-1}} T_{\gamma_{j1}} T_{\gamma_{j2}} \cdots$$

These monomials are not necessarily normalized (we may have to permute the factors to normalize them), but since $\gamma_{j1} < \alpha_i$ their normalizations are smaller lexicographically than M. Since M was the minimal counterexample, each such monomial is expressible modulo (2.5) as a linear combination of standard monomials. The same must then be true about M which gives us a contradiction. □

As a consequence of the discussion from Section 2.1 and the above theorem we obtain the following.

Corollary 2.10. *If A is an ASL over a reduced ring K then A is reduced.*

Proof. Assume first that A is a discrete ASL. It follows from Theorem 2.9 that

$$A \simeq K[T_\alpha : \alpha \in H]/(T_\alpha \cdot T_\beta : \alpha, \beta \,\text{incomparable}).$$

Since incomparable implies distinct, $T_\alpha \cdot T_\beta$ is a square-free monomial for every incomparable α, β, hence A is a reduced ring by Exercise 1(a).

For the general case we do induction on $\operatorname{ind}(A)$. By Theorem 2.8 we can find a multiplicative family of ideals $I_\bullet$ such that $\operatorname{gr}(A)$ is an ASL on the same poset H over the same ring K, and in addition $\operatorname{ind}(\operatorname{gr}(A)) < \operatorname{ind}(A)$. By induction we have that $\operatorname{gr}(A)$ is reduced, thus by Exercise 4 we conclude that A is also reduced. □

2.3. Exercises.

(1) (a) Verify that if K is a reduced ring and if $I \subset K[z_1, \cdots, z_n]$ is an ideal generated by square-free monomials then $K[z_1, \cdots, z_n]/I$ is a reduced ring.
(b) Show that if $I \subset K[z_1, \cdots, z_n]$ is an ideal generated by square-free monomials then the quotient ring $A = K[z_1, \cdots, z_n]/I$ can be given the structure of an ASL over K.

(2) Verify carefully that Example 2.3 gives rise to an ASL.

(3) Prove Lemma 2.5.

(4) Show that if $a, b \in A$ then in the associated graded ring $\operatorname{gr}(A)$ we either have that $\operatorname{in}(a) \cdot \operatorname{in}(b) = \operatorname{in}(ab)$ or $\operatorname{in}(a) \cdot \operatorname{in}(b) = 0$. Recalling the assumption (2.3) conclude that
- If $\operatorname{in}(a)$ is a non-zero divisor in $\operatorname{gr}(A)$ then a is a non-zero divisor in A. If $\operatorname{gr}(A)$ is a domain then A is also a domain.

- If a is nilpotent in A then $\mathrm{in}(a)$ is nilpotent in $\mathrm{gr}(A)$. If $\mathrm{gr}(A)$ is reduced then A is also reduced.

(5) Check that we have isomorphisms $\mathcal{R}/(t) \simeq \mathrm{gr}(A)$ and $\mathcal{R}/(t-1) \simeq A$.

(6) Check that if A is an ASL on a poset H over a ring K and if L is a K-algebra then $A \otimes_K L$ is an ASL on $H \otimes 1$ over L, where the poset structure on $H \otimes 1$ is that induced from the identification with H.

(7) Finish the proof of Theorem 2.7 by verifying condition (ASL-2) for the Rees algebra $\mathcal{R}$. More precisely, show that if $\tilde{\alpha}, \tilde{\beta}$ are incomparable then (with the notation of (2.1)) if we let

$$m_i = \mathrm{ord}_{I_\bullet}(\gamma_{i1} \cdot \gamma_{i2} \cdots)$$

then $m_i \geq \mathrm{ord}_{I_\bullet}(\alpha) + \mathrm{ord}_{I_\bullet}(\beta)$ for all i and

$$\tilde{\alpha} \cdot \tilde{\beta} = \sum_i c_i \cdot t^{m_i - \mathrm{ord}_{I_\bullet}(\alpha) - \mathrm{ord}_{I_\bullet}(\beta)} \cdot (\tilde{\gamma_{i1}} \cdot \tilde{\gamma_{i2}} \cdots)$$

(8) Prove the following strengthening of the straightening relations (ASL-2). If $M = \alpha_1 \alpha_2 \cdots \alpha_k$ is a non-standard monomial in A and if

$$M = \sum_i c_i \cdot (\gamma_{i1} \cdot \gamma_{i2} \cdots)$$

is the expression of M as a linear combination of standard monomials, then $\gamma_{i1} \leq \alpha_j$ for all i, j.

3. The Cohen–Macaulay property for graded ASLs

A **(positively) graded ring** is a ring R together with a decomposition

$$R = \bigoplus_{i \geq 0} R_i, \text{ such that } R_i \cdot R_j \subset R_{i+j} \text{ for all } i, j \geq 0.$$

A **graded R-module** is a module M with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $R_i \cdot M_j \subseteq M_{i+j}$ for all $i, j \geq 0$. If $R_0 = \kappa$ is a field then R has a unique maximal homogeneous ideal $\mathfrak{m} = \bigoplus_{i>0} R_i$: it follows from Exercise 3 that in order to prove that R is Cohen-Macaulay it is enough to check that $R_\mathfrak{m}$ is.

Suppose from now on that $R_0 = \kappa$ is a field, and let $d = \dim(R) = \dim(R_\mathfrak{m})$. A **homogeneous system of parameters** is a collection of homogeneous elements $x_1, \cdots, x_d$ with the property that the ideal $(x_1, \cdots, x_d)$ is $\mathfrak{m}$-primary, or equivalently satisfying

$$\dim(R/(x_1, \cdots, x_d)) = 0.$$

If $x_1, \cdots, x_d$ form a homogeneous system of parameters then R is a finite $k[x_1, \cdots, x_d]$-module (by the **graded Nakayama Lemma**), and therefore $x_1, \cdots, x_d$ are algebraically independent. Conversely, if R is finite over a subring $k[x_1, \cdots, x_d]$ generated by homogeneous elements $x_1, \cdots, x_d$ then these elements form a homogeneous system of parameters and in particular they are algebraically independent. We have the equivalence

$$R \text{ is CM } \iff x_1, \cdots, x_d \text{ form a regular sequence on } R \iff R \text{ is a free } \kappa[x_1, \cdots, x_d]\text{-module.}$$

Throughout this section we will assume that A is a **graded ASL** over a field κ: this means that in addition to (ASL-1) and (ASL-2) the ring A satisfies

(ASL-0) The ring A is graded, $A = \bigoplus_{i \geq 0} A_i$, with $A_0 = \kappa$, and the elements of H are homogeneous of positive degree.

Given an element $\alpha \in H$ we define its **rank** to be the length of a maximal descending chain in H starting at α: $\text{rank}(\alpha) = k$ if and only if there exists $\alpha = \alpha_k > \cdots > \alpha_1$, $\alpha_i \in H$, and k is maximal with this property. For a subset $U \subset H$ we define

$$\text{rank}(U) = \max\{\text{rank}(\alpha) : \alpha \in U\}.$$

In particular for the poset H in Example 2.3 we have $\text{rank}(x_{11}) = 2$ and $\text{rank}(H) = 4$.

Proposition 3.1. *Let A be a graded ASL on H over a field κ and let $\mathfrak{m} = (H)$ denote the maximal homogeneous ideal (which is generated by the elements of H). We have that*

$$\dim(A) = \text{height}(\mathfrak{m}) = \text{rank}(H).$$

Moreover, a homogeneous system of parameters can be constructed as follows. We let m_i denote the least common multiple of the degrees $\deg(\alpha)$ of elements $\alpha \in H$ with $\text{rank}(\alpha) = i$, and write $e(\alpha) = m_i / \deg(\alpha)$ if $\text{rank}(\alpha) = i$. The elements

$$x_i = \sum_{\alpha \in H,\ \text{rank}(\alpha)=i} \alpha^{e(\alpha)}, \quad i = 1, \cdots, r,$$

form a homogeneous system of parameters for A.

Proof. The equality $\dim(A) = \text{height}(\mathfrak{m})$ follows from the fact that $A_0 = \kappa$ is a field and $\mathfrak{m}$ is the maximal homogeneous ideal in A. Since A is a finitely generated κ-algebra, to prove that $\dim(A) \geq \text{rank}(H)$ it suffices to find a polynomial subring $P \subseteq A$ of dimension $r = \text{rank}(H)$. Consider a chain $\alpha_1 < \cdots < \alpha_r$ in H, which exists by the definition of $r = \text{rank}(H)$. Note that the monomials $\alpha_1^{i_1} \cdots \alpha_r^{i_r}$ $(i_1, \cdots, i_r \geq 0)$ are standard, hence by (ASL-1) they are linearly independent over κ. This means that $P = \kappa[\alpha_1, \cdots, \alpha_r]$ is a polynomial ring of dimension r contained in A.

Since $\deg(\alpha^{e(\alpha)}) = m_i$ if $\mathrm{rank}(\alpha) = i$, it is clear that the elements x_i are homogeneous. We prove by induction that they form a system of parameters. We let $H_1 \subset H$ denote the set of minimal elements in H (elements of rank 1), and let $\overline{H} = H \setminus H_1$ with its induced poset structure. We also consider the ideal $I_1 = (H_1)$ generated by the minimal elements. It is clear that $\mathrm{rank}(\overline{H}) = \mathrm{rank}(H) - 1$, and it follows from Exericise 5 that $\overline{A} = A/I_1$ is an ASL on $\overline{H}$ over κ (and it is clearly graded). By induction, the elements $\overline{x}_2, \cdots, \overline{x}_r$ form a system of parameters for $\overline{A}$, so $I_1 + (x_2, \cdots, x_r)$ is $\mathfrak{m}$-primary. To finish the induction we need to check that a sufficiently large power of every $\alpha \in H_1$ belongs to $(x_1, x_2, \cdots, x_r)$.

Consider then any $\alpha \in H_1$ and note that for every $\alpha \neq \beta \in H_1$ we have that α, β are incomparable, hence by (ASL-2) we obtain $\alpha \cdot \beta = 0$. This implies that

$$\alpha^{e(\alpha)+1} = \alpha^{e(\alpha)+1} + \sum_{\alpha \neq \beta \in H_1} \alpha \cdot \beta^{e(\beta)} = \alpha \cdot x_1 \in (x_1, x_2, \cdots, x_r)$$

which is what we wanted to prove. ∎

Corollary 3.2. *The Plücker algebra has dimension $m \cdot (n-m) + 1$.*

Proof. Any maximal chain in the Plücker poset has the form

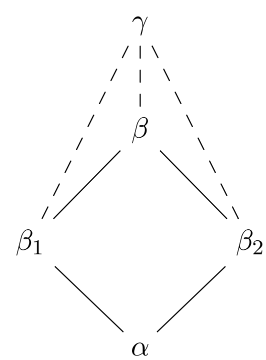
$$[1, 2, \cdots, m] < [1, 2, \cdots, m+1] < \cdots < [n-m+1, \cdots, n]$$

where two consecutive terms $[c_1, \cdots, c_m] < [d_1, \cdots, d_m]$ differ in precisely one component i, and for that component $d_i = c_i + 1$. It is then clear (see also Exercise 4) that the sum of the entries goes up by one at each step, so the length of such a chain is

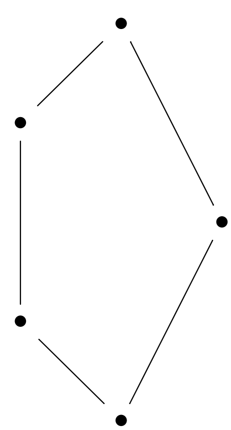
$$((n-m+1) + (n-m+2) + \cdots + n) - (1 + 2 + \cdots + m) + 1 = m \cdot (n-m) + 1.$$ ∎

We will give a combinatorial condition on the poset H which will guarantee the Cohen-Macaulayness of A, irrespective of the straightening relations! To do this, we need some definitions. An element $\beta \in H$ is a **cover** of $\alpha \in H$ if $\beta > \alpha$ and if there is no element γ lying strictly between α and β (i.e. satisfying $\beta > \gamma > \alpha$).

The poset H is said to be **wonderful** if the following holds after a smallest and a greatest element $-\infty$ and ∞ have been added to H: if $\alpha \in H \cup \{-\infty\}$, $\gamma \in H \cup \{\infty\}$, and $\beta_1, \beta_2 \in H$ are covers of α satisfying $\beta_1, \beta_2 < \gamma$ then there exists an element $\beta \in H \cup \{\infty\}$ with $\beta \leq \gamma$ which covers both β_1 and β_2. Pictorially, we have



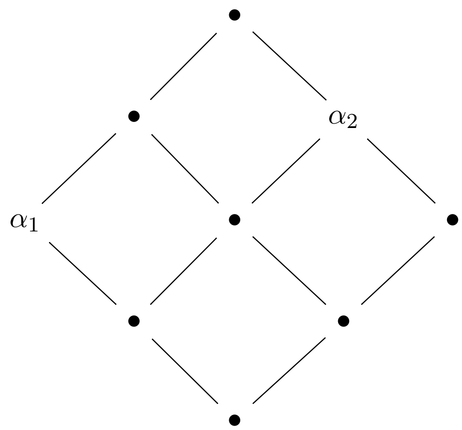
where filled lines correspond to covering relations, and dotted lines to pairs of ordered (not necessarily covering) elements. An example of a poset which is not wonderful is given by the following Hasse diagram:

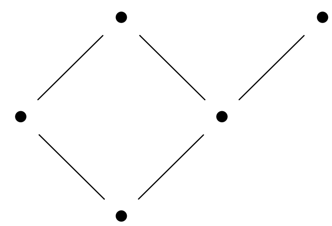


Equipped with the notion of a wonderful poset we will prove the following.

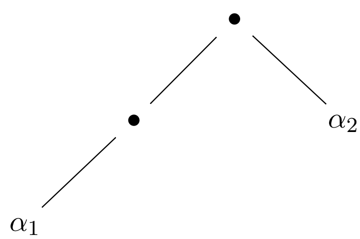
Theorem 3.3. *A graded graded ASL on a wonderful poset H over a field κ is Cohen-Macaulay.*

The proof of Theorem 3.3 will be an inductive argument based on a number of preliminary results which we establish next. We begin with a definition. If H is a poset then we define a **poset ideal** J in H to be a subset $J \subseteq H$ with the property that for every $\alpha \in J$ and every $\beta \leq \alpha$ we have $\beta \in J$. An example of a wonderful poset H and an ideal $J \subset H$ is given by

$H =$ 

$\supset J =$  (3.1)

(we singled out the elements α_1, α_2 for later use). Note that the subposet $H \setminus J$ is not wonderful:

$H \setminus J =$ 

However, we have the following criterion:

Lemma 3.4. *Suppose that $J \subset H$ is a poset ideal, and that for every minimal elements $\beta_1, \beta_2 \in H \setminus J$ and every $\gamma \in (H \setminus J) \cup \{\infty\}$ with $\beta_1, \beta_2 < \gamma$ there exists an element $\beta \in (H \setminus J) \cup \{\infty\}$ with $\beta \leq \gamma$ which covers both β_1, β_2. If H is wonderful then $H \setminus J$ is wonderful.*

Proof. Consider two elements $\beta_1, \beta_2 \in H \setminus J$ which cover an element $\alpha \in (H \setminus J) \cup \{-\infty\}$, and assume that $\gamma \in (H \setminus J) \cup \{\infty\}$ satisfies $\gamma > \beta_1, \beta_2$. We have to verify that there exists $\beta \in (H \setminus J) \cup \{\infty\}$, $\beta \leq \gamma$, which is a cover of both β_1, β_2.

If $\alpha = -\infty$ then β_1, β_2 have to be minimal elements of H by the definition of cover. The existence of β then follows from the hypothesis. We may then assume that $\alpha \in H \setminus J$. Since the poset H is wonderful, there exists $\beta \in H \cup \{\infty\}$, $\beta \leq \gamma$, which is a cover of both β_1, β_2. If $\beta \in J$ then $\beta \geq \beta_1 \geq \alpha$, and since J is a poset ideal it follows that $\alpha \in J$, a contradiction. This means that $\beta \in (H \setminus J) \cup \{\infty\}$, as desired. $\square$

Given a subset $\mathcal{A} \subset H$, the **poset ideal cogenerated by** $\mathcal{A}$ is defined to be the subset $J \subset H$ consisting of elements β with $\beta \not\geq \alpha$ for all $\alpha \in \mathcal{A}$ (check that this is indeed a poset ideal). Equivalently, $H \setminus J$ is the subposet of H consisting of elements β with $\beta \geq \alpha$ for some $\alpha \in \mathcal{A}$. In (3.1) the ideal J is cogenerated by $\{\alpha_1, \alpha_2\}$ and we saw that $H \setminus J$ is not wonderful. One can show by contrast that if J is a poset ideal cogenerated by a single element the $H \setminus J$ is wonderful (see Exercise 7). Also, when $\mathcal{A}$ consists of minimal elements we have the following.

Corollary 3.5. *If $J \subset H$ is the ideal cogenerated by a set of minimal elements $\alpha_1, \cdots, \alpha_t \in H$ and if H is wonderful then $H \setminus J$ is wonderful.*

Proof. We will verify that J satisfies the hypothesis of Lemma 3.4, and for that we note that the minimal elements of $H \setminus J$ are precisely $\alpha_1, \cdots, \alpha_t$. Suppose that $\gamma \in (H \setminus J) \cup \{\infty\}$ is such that $\gamma > \alpha_i, \alpha_j$ for some $i \neq j$. Since H is wonderful, we can find $\beta \in H \cup \{\infty\}$, $\beta \leq \gamma$, such that β covers α_i, α_j. Since J is cogenerated by $\alpha_1, \cdots, \alpha_t$, it follows that $\beta \in (H \setminus J) \cup \{\infty\}$, so Lemma 3.4 applies. □

We will also need the following commutative algebra fact.

Lemma 3.6. *Write $r = \dim(A)$. Assume that $I, I' \subset A$ are homogeneous ideals with $I \cap I' = 0$, that $\dim(A/I) = \dim(A/I') = r$ and that $\dim(A/(I + I')) = r - 1$. If $A/I, A/I'$ and $A/(I + I')$ are Cohen-Macaulay then A is also Cohen-Macaulay.*

Proof. We have an exact sequence

$$0 \longrightarrow A/(I \cap I') \longrightarrow A/I \oplus A/I' \longrightarrow A/(I + I') \longrightarrow 0$$

and $A/(I \cap I') = A$ by the hypothesis. Since $A/I, A/I'$ are Cohen-Macaulay of dimension r, $\text{depth}(A/I) = \text{depth}(A/I') = r$. Since $A/(I + I')$ is Cohen-Macaulay of dimension $r - 1$, $\text{depth}(A/(I + I')) = r - 1$. By the depth lemma (in Ryo's lectures) we conclude that $\text{depth}(A) \geq r$, i.e. A is Cohen-Macaulay. □

Proof of Theorem 3.3. We do induction on $|H|$, and write $r = \text{rank}(H) = \dim(A)$ by Proposition 3.1. If H contains a unique minimal element α, then α is a non-zero divisor on A and the poset $\overline{H} = H \setminus \{\alpha\}$ is wonderful with $\text{rank}(\overline{H}) = r - 1$ (see Exercise 6). By Exercise 5, $\overline{A} = A/(\alpha)$ is an ASL on $\overline{H}$. It follows by induction that $\overline{A}$ is Cohen-Macaulay, and by Proposition 3.1 that $\dim(\overline{A}) = r - 1$. This is enough to conclude that A is Cohen-Macaulay.

Assume now that H has at least two minimal elements, namely $\alpha_1, \cdots, \alpha_t$, $t \geq 2$. We will prove that A is Cohen-Macaulay by applying Lemma 3.6. Let $J \subset H$ be the poset ideal cogenerated by α_1, and let $J' \subset H$ be the poset ideal cogenerated by $\alpha_2, \cdots, \alpha_t$. Let $I \subset A$ be the ideal generated by J, and $I' \subset A$ the ideal generated by J'. By Exercise 5, A/I is an ASL on $H \setminus J$, and A/I' is an ASL on $H \setminus J'$. By Corollary 3.5, $H \setminus J$ and $H \setminus J'$ are wonderful, thus A/I and A/I' are Cohen-Macaulay by induction. To show that $\dim(A/I) = \dim(A/I') = r$ it suffices to show that $H \setminus J$ and $H \setminus J'$ contain chains of length r. Consider a maximal chain in H containing α_1: by Exercise 7 this chain has length r, and since

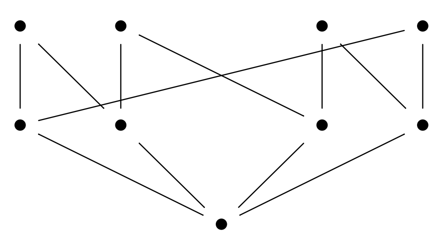
J is cogenerated by α_1 this chain is contained in $H \setminus J$, so $\mathrm{rank}(H \setminus J) = r$. Replacing α_1 with α_2 and repeating the argument we get $\mathrm{rank}(H \setminus J') = r$.

Using Exercise 5, the intersection $I \cap I'$ is spanned by standard monomials $M = \gamma_1 \cdot \gamma_2 \cdots$ which contain a factor $\beta \in J$ and a factor $\beta' \in J'$, i.e. $\beta \not\geq \alpha_1$ and $\beta' \not\geq \alpha_i$ for $i = 2, \cdots, t$. Since $\gamma_1 \in H$ we get $\gamma_1 \geq \alpha_j$ for some j. Since the monomial M is standard we get $\gamma_1 \leq \beta, \beta'$. This shows that $\alpha_j \leq \beta, \beta'$, which is impossible, so $I \cap I' = 0$.

To finish the proof we have to show that $A/(I + I')$ is Cohen-Macaulay of dimension $r-1$. Since $I + I'$ is the ideal generated by $J \cup J'$, it suffices to show that $H \setminus (J \cup J')$ is a wonderful poset of rank $r-1$. We first verify that the rank is $r-1$. Note that $H \setminus (J \cup J') = (H \setminus J) \cap (H \setminus J')$ consists of elements β with $\beta \geq \alpha_1$ and $\beta \geq \alpha_i$ for some $i > 1$. It follows that $H \setminus (J \cup J')$ contains no minimal elements of H, so $\mathrm{rank}(H \setminus (J \cup J')) \leq r - 1$. Moreover, since α_1, α_2 cover $-\infty$ and since H is wonderful we can find $\beta \in H$ which covers both α_1, α_2. Consider a maximal chain $\alpha_1 < \beta < \gamma_1 < \cdots < \gamma_{r-2}$ in H (using Exercise 7), and note that since $\gamma_i > \beta > \alpha_1, \alpha_2$ we have $\beta, \gamma_i \in H \setminus (J \cup J')$, i.e. $\beta < \gamma_1 < \cdots < \gamma_{r-2}$ is a chain of length $r-1$ in $H \setminus (J \cup J')$ and therefore $\mathrm{rank}(H \setminus (J \cup J')) = r - 1$.

Finally, we show that $H \setminus (J \cup J')$ is wonderful, for which we apply Lemma 3.4. To do that we first show that if β is minimal in $H \setminus (J \cup J')$ then β covers α_1. We know that $\beta \geq \alpha_1$ and $\beta \geq \alpha_i$ for some $i > 1$, and α_1, α_i cover $-\infty$ since they are minimal elements of H. Since H is wonderful, we can find $\beta' \in H$ with $\beta' \leq \beta$ such that β' covers α_1, α_i, so in particular $\beta' \in H \setminus (J \cup J')$. But since β is minimal in $H \setminus (J \cup J')$ we must have $\beta' = \beta$ and thus β covers α_1. To check the hypothesis of Lemma 3.4 we consider minimal elements $\beta_1, \beta_2 \in H \setminus (J \cup J')$, and an element $\gamma \in (H \setminus (J \cup J')) \cup \{\infty\}$ with $\gamma > \beta_1, \beta_2$. Since β_1, β_2 cover α_1 and H is wonderful, there exists $\beta \in H \cup \{\infty\}$ with $\beta \leq \gamma$ which covers both β_1, β_2. This implies that $\beta \geq \alpha_1$ and $\beta \geq \alpha_i$ for some $i > 1$, so $\beta \in (H \setminus (J \cup J')) \cup \{\infty\}$ and Lemma 3.4 applies, concluding our proof. $\square$

The condition that H is wonderful is by no means necessary for the Cohen-Macaulayness of an ASL on H. For instance, [HW85, Fig. 4] gives a poset H which is not wonderful, but for which there exists a Gorenstein graded ASL domain on H:



The following question is unresolved.

Open Question 3.7 (Watanabe)**.** If A is a graded ASL domain over a field of characteristic zero, is it true that A is Cohen-Macaulay?

When $\dim(A) = 3$ this question has an affirmative answer, as explained in [HW85, Section 2].

3.1. **Exercises.**

(1) Suppose that A is a graded ASL over a field κ. Show that if $\dim(A) = 1$ then

$$A \simeq \kappa[x_1, \cdots, x_n]/(x_i \cdot x_j : i \neq j).$$

(2) Suppose that A, A' are graded ASLs on the posets H, H' over a field κ, and consider the **Segre product**

$$A\#A' = \bigoplus_{i \geq 0} A_i \otimes_\kappa A'_i.$$

Show that $A\#A'$ is a graded ASL on the **product poset** $H \times H'$ (where the order is defined by $(\alpha, \alpha') \leq (\beta, \beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$).

(3) Let R be a Noetherian graded ring and let M be a finitely generated graded R-module. Given an arbitrary ideal $I \subset R$ we define a corresponding **graded module** I^* to be the ideal generated by all homogeneous elements $a \in I$. Show that

(a) For $\mathfrak{p} \in \operatorname{Spec}(R)$ the localization $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $M_{\mathfrak{p}^*}$ is.

(b) M is Cohen-Macaulay if and only if $M_\mathfrak{p}$ is for all graded prime ideals $\mathfrak{p}$.

(c) Suppose that $R_0 = \kappa$ is a field and let $\mathfrak{m} = \bigoplus_{i>0} R_i$ denote the maximal homogeneous ideal. Show that M is Cohen-Macaulay if and only if $M_\mathfrak{m}$ is.

(4) Show that the Plücker poset in Example 2.4 is wonderful and conclude that the Plücker algebra is Cohen-Macaulay. Verify that for an element $[c_1, \cdots, c_m]$ in the Plücker poset we have

$$\operatorname{rank}([c_1, \cdots, c_m]) = (c_1 - 1) + (c_2 - 2) + \cdots + (c_m - m) + 1.$$

(5) Let A be an ASL on H over K, let $J \subseteq H$ be a poset ideal, and let $I(J) \subset A$ denote the ideal generated by the elements of J.

(a) Prove that a standard monomial is contained in $I(J)$ if and only if it contains a factor in J, and conclude that $I(J)$ is spanned over K by the standard monomials it contains.

(b) Show that $A/I(J)$ is an ASL on $H \setminus J$ over K.

(6) Suppose that A is an ASL on a poset H containing a unique minimal element α, and let $\overline{H} = H \setminus \{\alpha\}$. Show that $\operatorname{rank}(\overline{H}) = \operatorname{rank}(H) - 1$ and that α is a non-zero divisor on A. Show also that if H is wonderful then $\overline{H}$ is wonderful.

(7) Suppose that H is a wonderful poset.

(a) Show that if J is a poset ideal cogenerated by a single element $\alpha \in H$ then $H \setminus J$ is wonderful.

(b) Show that the length of any maximal chain in H is equal to rank(H).

(8) Consider the following variation of the Plücker poset. Fix $1 \leq m \leq n$ and let

$$H = \{(i_1, \cdots, i_r | j_1, \cdots, j_r) : 1 \leq r \leq m,\ 1 \leq i_1 < i_2 < \cdots < i_r \leq m,\ 1 \leq j_1 < j_2 < \cdots < j_r \leq n\}$$

with the partial order defined by

$$(i_1, \cdots, i_r | j_1, \cdots, j_r) \leq (i'_1, \cdots, i'_s | j'_1, \cdots, j'_s) \text{ if and only if}$$
$$r \geq s \text{ and } i_k \leq i'_k,\ j_k \leq j'_k \text{ for } k = 1, \cdots, s.$$

Show that:

(a) When $m = n = 2$, H is the poset from Example 2.3.

(b) For any $m \leq n$ the poset H is wonderful.

(c) Fix $1 \leq r \leq m$ and consider the subset $J_r \subset H$ consisting of all elements $(i_1, \cdots, i_s | j_1, \cdots, j_s)$ with $s \geq r$. Show that J_r is a poset ideal cogenerated by a single element of H and conclude that $H \setminus J_r$ is a wonderful poset.

4. Determinantal rings

In this section we consider the polynomial ring $A = \kappa[x_{ij}]$ on the entries of the generic $m \times n$ matrix X. The goal is to put on A the structure of a graded ASL over κ which will help us study questions about determinantal ideals (of course there is an easy ASL structure given in Example 2.1, but that will not be helpful for us). Given $1 \leq a_1, \cdots, a_r \leq m$ and $1 \leq b_1, \cdots, b_r \leq n$ we consider the corresponding $r \times r$ minor

$$[a_1, \cdots, a_r | b_1, \cdots, b_r] = \det(x_{a_i b_j})_{1 \leq i,j \leq r},$$

and note that just as in the case of Plücker coordinates the formation of this minor is skew-symmetric in a_i's (resp. b_j's), and is zero when repetitions occur among the a_i's (resp. b_j's). We define H to be the poset of all minors $[a_1, \cdots, a_r | b_1, \cdots, b_r]$ with $a_1 < \cdots < a_r$, $b_1 < \cdots < b_r$, with the order defined (as in Exercise 8 of Section 3) by

$$[a_1, \cdots, a_r | b_1, \cdots, b_r] \leq [a'_1, \cdots, a'_s | b'_1, \cdots, b'_s] \text{ if and only if}$$
$$r \geq s \text{ and } a_k \leq a'_k,\ b_k \leq b'_k \text{ for } k = 1, \cdots, s.$$

The main result of this section is the following.

Theorem 4.1. *The polynomial ring A is a graded ASL on H over κ.*

As a consequence, we draw similar conclusions about the determinantal rings obtained as quotients of A by determinantal ideals. For $t = 1, \cdots, m$ we will denote by I_t the ideal generated by the $t \times t$ minors of X. If we define the poset ideals $J_t \subset H$ as in Exercise 8 from Section 3 then we obtain the following.

Corollary 4.2. *For every $t = 1, \cdots, m$ the quotient ring A/I_t is a graded ASL on $H \setminus J_t$ over κ, and it is Cohen-Macaulay of dimension $(t-1) \cdot (m+n-t+1)$.*

Proof. We have that I_t is the ideal of A generated by the elements of J_t, thus by Exercise 5 in Section 3 we get that A/I_t is an ASL on $H \setminus J_t$ over κ (and is clearly graded). By Exercise 8 in Section 3, the poset $H \setminus J_t$ is wonderful, hence A/I_t is Cohen-Macaulay. You will verify the assertion about the dimension of A/I_t in Exercise 8. □

To prove Theorem 4.1, we relate A to the Plücker algebra of a (slightly larger) generic matrix. We let $\tilde{X}$ denote the $m \times (n+m)$ generic matrix

$$\tilde{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} & x_{1,n+1} & \cdots & x_{1,n+m-1} & x_{1,n+m} \\ x_{21} & x_{22} & \cdots & x_{2n} & x_{2,n+1} & \cdots & x_{2,n+m-1} & x_{2,n+m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & x_{m,n+1} & \cdots & x_{m,n+m-1} & x_{m,n+m} \end{bmatrix}$$

and let $\tilde{R}$ denote the Plücker algebra generated by the maximal minors of $\tilde{X}$. We let $\tilde{H}$ denote the corresponding Plücker poset, and note that $\tilde{H}$ has a unique maximal element, namely $[n+1, \cdots, n+m]$. We consider the extended matrix

$$X^e = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 0 & \cdots & 0 & 1 \\ x_{21} & x_{22} & \cdots & x_{2n} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & 1 & \cdots & 0 & 0 \end{bmatrix}$$

and note that the natural specialization map $\tilde{X} \longrightarrow X^e$ induces a ring homomorphism

$$\phi : \tilde{R} \longrightarrow A$$

which is surjective by Exercise 1 in Section 1.

Lemma 4.3. *Given $1 \leq t \leq m$, $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$ we define a sequence $n+1 \leq b_{t+1} < \cdots < b_m \leq m+n$ by the equality*

$$\{a_1, \cdots, a_t, m+n+1-b_m, \cdots, m+n+1-b_{t+1}\} = \{1, \cdots, m\}. \tag{4.1}$$

We have that

$$[a_1,\cdots,a_t|b_1,\cdots,b_t]=\pm\phi([b_1,\cdots,b_m])$$

and the association

$$[a_1,\cdots,a_t|b_1,\cdots,b_t]\longrightarrow[b_1,\cdots,b_m] \tag{4.2}$$

establishes an isomorphism of posets between H *and* $\tilde{H}\setminus\{[n+1,\cdots,n+m]\}$.

Example 4.4. Consider the case when $m=3$ and $n=4$. Equation (4.2) gives a correspondence

$$[1,2,3|1,2,3]\longleftrightarrow[1,2,3],\quad [2,3|2,4]\longleftrightarrow[2,4,7],\quad [3|2]\longleftrightarrow[2,6,7].$$

We have

$$[1,2,3|1,2,3]\leq[2,3|2,4]\leq[3|2] \text{ in } H \text{ and } [1,2,3]\leq[2,4,7]\leq[2,6,7] \text{ in } \tilde{H}.$$

It is more convenient to define $\overline{i}=m+n+1-i$ for $i=1,\cdots,m$, so that the columns of $\tilde{X}$ are labeled $1,2,\cdots,n,\overline{m},\overline{m-1},\cdots,\overline{1}$. The Plücker coordinate $[c_1,\cdots,c_t,\overline{d}_1,\cdots,\overline{d}_{m-t}]$ then corresponds to the $t\times t$ minor of X determined by the columns $c_1,\cdots,c_t$ and the rows different from $d_1,\cdots,d_{m-t}$. The correspondence between minors and Plücker coordinates in the example above is then given by

$$[1,2,3|1,2,3]\longleftrightarrow[1,2,3],\quad [2,3|2,4]\longleftrightarrow[2,4,\overline{1}],\quad [3|2]\longleftrightarrow[2,\overline{2},\overline{1}].$$

Proof of Lemma 4.3. It is easy to see that (4.2) gives a bijection between H and $\tilde{H}\setminus\{[n+1,\cdots,n+m]\}$: to define the inverse of (4.2), for a given $[b_1,\cdots,b_m]\in\tilde{H}\setminus\{[n+1,\cdots,n+m]\}$ we let t be the unique index for which $b_t\leq n$, $b_{t+1}\geq n+1$ and define $a_1,\cdots,a_t$ by (4.1). We then only need to verify that this bijection identifies the orderings of the two posets. This is the content of Exercise 1. ☐

Proof of Theorem 4.1. We write

$$\epsilon=(-1)^{\binom{n}{2}}=\det\begin{bmatrix}0&\cdots&0&1\\0&\cdots&1&0\\\vdots&\iddots&\vdots&\vdots\\1&\cdots&0&0\end{bmatrix}$$

and note that $\phi([n+1,\cdots,n+m])=\epsilon$, so ϕ induces a surjective ring homomorphism

$$\overline{\phi}:\tilde{R}/([n+1,\cdots,n+m]-\epsilon)\longrightarrow A.$$

By Corollary 3.2 we have $\dim(\tilde{R})=m\cdot n+1$, while $\dim(A)=m\cdot n$ since A is a polynomial ring in $m\cdot n$ variables. By Exercise 2, the quotient $\tilde{R}/([n+1,\cdots,n+m]-\epsilon)$ is an integral domain (necessarily of dimension smaller than that of $\tilde{R}$), hence $\overline{\phi}$ is in fact an isomorphism.

By Lemma 4.3, ϕ maps standard monomials in $\tilde{R}$ to standard monomials in A, so the algebra A is spanned by standard monomials. Moreover, since $[n+1,\cdots,n+m]$ is the unique maximal element of $\tilde{H}$ and is mapped to a constant by ϕ, it is clear that the straightening relations of A are induced by those of $\tilde{R}$ and (ASL-2) holds for A. To conclude that A is an ASL on H we have to verify that the standard monomials are linearly independent.

By Lemma 4.3 the map ϕ induces a bijection between standard monomials in $\tilde{R}$ not containing $[n+1,\cdots,n+m]$ as a factor, and standard monomials in A. We let $\tilde{A} \subset R$ denote the span of standard monomials in $\tilde{R}$ not involving $[n+1,\cdots,n+m]$. Since $\overline{\phi}$ is an isomorphism, to prove the linear independence of standard monomials in A it suffices to check that

$$\tilde{A} \cap ([n+1,\cdots,n+m]-\epsilon)\tilde{R} = 0. \tag{4.3}$$

Suppose that we have a relation

$$\begin{aligned}\sum a_i \cdot M_i &= ([n+1,\cdots,n+m]-\epsilon)\cdot \sum b_j \cdot N_j \\ &= \sum b_j \cdot (N_j \cdot [n+1,\cdots,n+m]) - \sum (b_j \cdot \epsilon)\cdot N_j\end{aligned}$$

where M_i are standard monomials not involving $[n+1,\cdots,n+m]$, while N_j are arbitrary standard monomials. Since $[n+1,\cdots,n+m]$ is the maximal element of $\tilde{H}$, all the monomials $N_j \cdot [n+1,\cdots,n+m]$ are standard. By assumption, the monomials M_i, N_j are also standard, hence the above relation can only hold if $a_i = b_j = 0$ for all i,j. This shows (4.3) and concludes our proof. □

To obtain a pictorial representation of the elements of H we define a **double (or bi)tableau** to be a pair $D = [T|T']$ where $|T| = |T'|$ (recall that this notation means that T and T' have the same shape, i.e. the same underlying Young diagram). The **shape** $|D|$ of is defined to be the partition $|T| = |T'|$. We say that D is a **standard double tableau** if both T, T' are standard. We represent a product of minors of X as a double tableau $[T, T']$ where the i-th minor in the product is given by

$$[T(i,1), T(i,2), \cdots | T'(i,1), T'(i,2), \cdots]$$

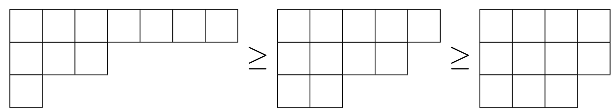
With these conventions, the standard monomials in $A = \kappa[x_{ij}]$ correspond precisely to standard double tableaux. For esthetic reasons, when picturing $[T|T']$ as a pair of filled Young diagrams we will draw instead of T a mirror image of T with respect to a vertical axis: for instance the standard monomial which is the product of minors from Example 4.4 is represented by the standard double tableau

$$[T|T'] = \begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 2 & 3 \\ & 3 & 2 & 2 & 4 & \\ & & 3 & 3 & & \end{array}, \text{ where } T = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{array}, \text{ and } T' = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & \\ 3 & & \end{array}.$$

Given two partitions λ, μ of the same size, we say that λ **dominates** μ and write $\lambda \geq \mu$ if for every $i \geq 1$ we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i.$$

For instance

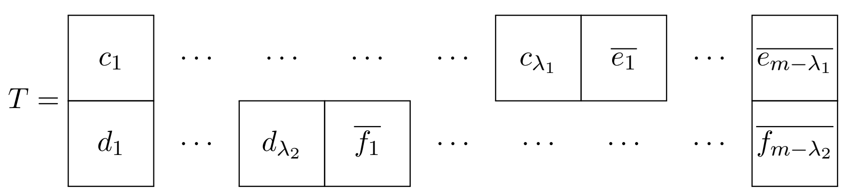


Note that the dominance relation is different from the **containment** relation $\lambda \supseteq \mu$, which means $\lambda_i \geq \mu_i$ for all i. It is always the case that if $\lambda \supseteq \mu$ then $\lambda \geq \mu$, but the converse fails in general as can be seen in the example above.

Proposition 4.5. *If D is a double tableau of shape λ and if $D = \sum a_i D_i$ is the representation of D as a linear combination of standard double tableaux, then $|D| \leq |D_i|$ for all i.*

Proof. Let D be a double tableau, and consider the corresponding tableau T with entries in the set $\{1, 2, \cdots, n, \overline{m}, \cdots, \overline{1}\}$ and constructed via the correspondence (4.1) with the conventions in Example 4.4. We apply the procedure from Lemma 1.7 to write T as a linear combination of standard tableaux, and then translate the results back to relations involving double tableaux. It is enough to analyze what happens for (double) tableaux with two rows.

Suppose that D is a double tableau of shape (λ_1, λ_2) and consider the corresponding tableau $T = [c_1, \cdots, c_m] \cdot [d_1, \cdots, d_m]$ with $c_i, d_j \in \{1, \cdots, n, \overline{m}, \cdots, \overline{1}\}$, and (at the expense of possibly changing the sign) $c_1 < \cdots < c_m$, $d_1 < \cdots < d_m$. We write $\overline{e_i} = c_{\lambda_1 + i}$ and $\overline{f_j} = d_{\lambda_2 + j}$, so that



has entries satisfying $c_1, \cdots, c_{\lambda_1}, d_1, \cdots, d_{\lambda_2} \in [n]$ and $e_1, \cdots, e_{m-\lambda_1}, f_1, \cdots, f_{m-\lambda_2} \in [m]$. To apply the Plücker relation (1.4) we consider the first index k such that $c_k \leq d_k$ and $c_{k+1} > d_{k+1}$. Since $i < \overline{j}$ for all $i \in [n]$ and $j \in [m]$, we see that either $k < \lambda_2$ or $k \geq \lambda_1$. In either case, we see that the relation (1.4) allows us to rewrite T as a linear combination of tableaux T_i such that each T_i has at least λ_1 entries from $\{1, \cdots, n\}$ in its first row, i.e. such that the corresponding double tableau D_i has shape $\mu = (\mu_1, \mu_2)$ with $\mu_1 \geq \lambda_1$ (and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$). Since any such μ satisfies $\mu \geq \lambda$, the conclusion follows. $\square$

For a partition λ with $\lambda_1 \leq m$ we let $A_\lambda \subset A$ denote the span of all double tableaux D of shape $|D| \geq \lambda$,

$$A_\lambda = \sum_{|D| \geq \lambda} \kappa \cdot D \tag{4.4}$$

and let $I_\lambda \subset A$ denote the span of all standard double tableaux D of shape $|D| \supseteq \lambda$,

$$I_\lambda = \sum_{|D| \supseteq \lambda} \kappa \cdot D \tag{4.5}$$

It follows from Proposition 4.5 that A_λ is an ideal in A, and you will verify in Exercise 5 that I_λ is an ideal as well. See [DCEP80] for an extensive study of these ideals.

4.1. **Exercises.**

(1) Verify that the bijection in Lemma 4.3 is an isomorphism of posets.

(2) Let R be a $\mathbb{Z}$-graded ring, let $0 \neq f \in R_1$, and let $u \in R_0$ be an invertible element (a unit). Show that the localization R_f is also $\mathbb{Z}$-graded, and that if we write $(R_f)_0$ for the degree 0 part of R_f then we have an isomorphism

$$R/(f - u) \simeq (R_f)_0.$$

In particular, if R is a domain then the same is true about $R/(f - u)$.

(3) Suppose that $m = n = 4$. Write the double tableau

$$\begin{array}{|c|c||c|c|} \hline 2 & 1 & 1 & 4 \\ \hline 4 & 3 & 2 & 3 \\ \hline \end{array}$$

as a linear combination of standard double tableaux. Check your work!

(4) (a) Let B be a normal domain, and let G be a subgroup of the group $\mathrm{Aut}(B)$ of automorphisms of B. Show that B^G is a normal domain.
(b) Assume that κ is an infinite field. Show that the Plücker algebra is the ring of invariants for the action of $\mathrm{SL}_m(\kappa)$ on the polynomial ring $A = \kappa[x_{ij}]$ and conclude that it is normal.

(5) Assume that κ is a field of characteristic zero.
(a) If λ is any partition with $\lambda_1 \leq m$ show that I_λ is an ideal in $A = \kappa[x_{ij}]$.
(b) Let $Z_{<t}$ denote the set of $m \times n$ matrices of rank $< t$ with entries in κ. Show that the ideal of polynomials $f \in A$ vanishing on $Z_{<t}$ is the same as the ideal I_t generated by the $t \times t$ minors of the generic matrix $X = (x_{ij})$.
(c) Show that $\sqrt{I_\lambda} = I_{\lambda_1}$ (using part (b)).

(6) Assume that κ is a field of characteristic zero. Show that the powers of the ideals I_t have a basis consisting of standard double tableaux, and using notation (4.4) that

$$I_t^d = A_{(t^d)},\ \text{where } (t^d) = (t, t, \cdots, t) \text{ is the partition with } d \text{ parts equal to } t.$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, write $\#\lambda = \lambda_1 + \lambda_2 + \cdots$ for the size of λ. Show that

$$I_t^d = \sum_{\substack{\#\lambda = t\cdot d \\ \lambda_1 \leq m,\ \lambda_{d+1}=0}} I_\lambda.$$

(7) Show that the ideal I_t can be generated up to radical by $m \cdot n - t^2 + 1$ elements.

(8) Show that $\dim(A/I_t) = (t-1)\cdot(m+n-t+1)$ in two ways:

(a) By studying the Hilbert function of A/I_t.

(b) By computing the rank of the poset $H \setminus J_t$.

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