Day 1

Let k be an algebraically closed field of characteristic $p > 0$. Given a finite module M over a Noetherian ring R, let $\mu_R(M)$ denote the minimal number of generators for M.

1. Let $\mathfrak{a}$ be a nonzero ideal of a Noetherian ring R of prime characteristic p. Let $f_1, \ldots, f_r \in R$ be a system of generators for $\mathfrak{a}$. For every integer $e \geq 1$, the ideal $\mathfrak{a}^{[p^e]} \subset R$ is defined as the ideal $(f_1^{p^e}, \ldots, f_r^{p^e})$. Verify that this definition of $\mathfrak{a}^{[p^e]}$ is independent of the choice of $f_1, \ldots, f_r$.

2. Let $R = k[[x, y, z]]/(xy + z^2)$.
   (a) Verify that $R^{1/p}$ is a maximal Cohen-Macaulay R-module.
   (b) Compute $\operatorname{rank}_R R^{1/p}$ and $\mu_R(R^{1/p})$.
   (c) Decompose the R-module $R^{1/p}$ into indecomposable R-modules (Hint: there are only two indecomposable maximal Cohen-Macaulay R-modules, R and (x, z), up to isomorphism).

3. The following is the complete list of rational double points $k[[x, y, z]]/(f)$ over k due to M. Artin. Determine which are F-pure and which are not.
   (A) $f = z^{n+1} + xy \ \ (n \geq 1)$
   $(D\text{-}1)$ $f = z^2 + x^2y + xy^n \ \ (n \geq 2,\ p = 2)$
   $(D\text{-}2)$ $f = z^2 + x^2y + xy^n + xy^{n-r}z \ \ (n \geq 2,\ n-1 \geq r \geq 1,\ p = 2)$
   $(D\text{-}3)$ $f = z^2 + x^2y + y^nz \ \ (n \geq 2,\ p = 2)$
   $(D\text{-}4)$ $f = z^2 + x^2y + y^nz + xy^{n-r}z \ \ (n \geq 2,\ n-1 \geq r \geq 1,\ p = 2)$
   $(D\text{-}5)$ $f = z^2 + x^2y + y^{n-1} \ \ (n \geq 4)$
   $(E_6\text{-}1)$ $f = z^2 + x^3 + y^2z \ \ (p = 2)$
   $(E_6\text{-}2)$ $f = z^2 + x^3 + y^2z + xyz \ \ (p = 2)$
   $(E_6\text{-}3)$ $f = z^2 + x^3 + y^4 \ \ (p \geq 3)$
   $(E_6\text{-}4)$ $f = z^2 + x^3 + y^4 + x^2y^2 \ \ (p = 3)$
   $(E_7\text{-}1)$ $f = z^2 + x^3 + xy^3$
   $(E_7\text{-}2)$ $f = z^2 + x^3 + xy^3 + x^2yz \ \ (p = 2)$
   $(E_7\text{-}3)$ $f = z^2 + x^3 + xy^3 + y^3z \ \ (p = 2)$
   $(E_7\text{-}4)$ $f = z^2 + x^3 + xy^3 + xyz \ \ (p = 2)$
   $(E_7\text{-}5)$ $f = z^2 + x^3 + xy^3 + x^2y^2 \ \ (p = 3)$
   $(E_8\text{-}1)$ $f = z^2 + x^3 + y^5$
   $(E_8\text{-}2)$ $f = z^2 + x^3 + y^5 + xy^3z \ \ (p = 2)$
   $(E_8\text{-}3)$ $f = z^2 + x^3 + y^5 + xy^2z \ \ (p = 2)$
   $(E_8\text{-}4)$ $f = z^2 + x^3 + y^5 + y^3z \ \ (p = 2)$
   $(E_8\text{-}5)$ $f = z^2 + x^3 + y^5 + xyz \ \ (p = 2)$
   $(E_8\text{-}6)$ $f = z^2 + x^3 + y^5 + x^2y^3 \ \ (p = 3)$
   $(E_8\text{-}7)$ $f = z^2 + x^3 + y^5 + x^2y^2 \ \ (p = 3)$
   $(E_8\text{-}8)$ $f = z^2 + x^3 + y^5 + xy^4 \ \ (p = 5)$

Day 2

- Let k be an algebraically closed field of characteristic $p > 0$.
- When A is a Noetherian reduced ring (but not necessarily a domain) of characteristic $p > 0$ with minimal primes $\mathfrak{p}_1 \dots, \mathfrak{p}_r$, we put
$$A^{1/p^e} = \{x \in \overline{Q(A/\mathfrak{p}_1)} \times \cdots \times \overline{Q(A/\mathfrak{p}_r)} \mid x^{p^e} \in A\}$$
for each integer $e \geq 1$. We say that A is F-finite if the natural inclusion $A \hookrightarrow A^{1/p}$ is a finite ring homomorphism. When A is F-finite, we say that A is F-pure if $A \hookrightarrow A^{1/p}$ splits as an A-module homomorphism. If, in addition, A is local, then the F-pure thresholds are defined as in the domain case.
- When A is an F-finite F-pure (but not necessarily local) domain of characteristic $p > 0$ and f is a nonzero non-unit element of A, we define the F-pure threshold $\mathrm{fpt}(f)$ by $\mathrm{fpt}(f) = \lim_{e\to\infty} \nu_e(f)/p^e$, where $\nu_e(f)$ is the largest nonnegative integer r such that the evaluation map
$$\mathrm{Hom}_A(A^{1/p^e}, A) \cdot f^{r/p^e} \to A$$
is surjective. As in the local case, we say that A is strongly F-regular if $\mathrm{fpt}(f) > 0$ for every nonzero non-unit $f \in A$.

4. Suppose that R is an F-finite Noetherian domain of characteristic $p > 0$ such that the localization $R_\mathfrak{m}$ is strongly F-regular for every maximal ideal $\mathfrak{m}$ of R.
   (a) Fix a nonzero element $f \in R$. For each $e \geq 1$, let $I_e(f)$ be the image of the R-linear map $\mathrm{Hom}_R(R^{1/p^e}, R) \to R$ sending φ to $\varphi(f^{1/p^e})$. Show that $I_e(f) \subset I_{e+1}(f)$ for each $e \geq 1$.
   (b) Show that R is strongly F-regular (Hint: the ascending chain $I_1(f) \subset I_2(f) \subset \cdots$ stabilizes to some ideal $I(f)$. Suppose, for a contradiction, that $I(f)$ is a proper ideal).

5. In the list of Exercise 3, determine which are strongly F-regular.

6. Let $(R, \mathfrak{m})$ be a d-dimensional F-finite F-pure local domain and $\mathfrak{a}, \mathfrak{b}$ be nonzero proper ideals of R. Verify the following properties.
   (a) $\mathrm{fpt}(\mathfrak{a}^n) = \mathrm{fpt}(\mathfrak{a})/n$ for each integer $n \geq 1$.
   (b) $\mathrm{fpt}(\mathfrak{a}R_\mathfrak{p}) \geq \mathrm{fpt}(\mathfrak{a})$ for every prime ideal $\mathfrak{p} \subset R$ containing $\mathfrak{a}$.
   (c) $\mathrm{fpt}(\mathfrak{a} + \mathfrak{b}) \leq \mathrm{fpt}(\mathfrak{a}) + \mathrm{fpt}(\mathfrak{b})$.
   (d) Let r be the vanishing order of $\mathfrak{a}$, that is, $r = \max\{n \geq \mathbb{Z}_{\geq 0} \mid \mathfrak{a} \subset \mathfrak{m}^n\}$. When R is a regular local ring, one has
$$\frac{1}{r} \leq \mathrm{fpt}(\mathfrak{a}) \leq \frac{d}{r}$$
(Hint: we may assume by Exercise 7 (a) that $R = k[[x_1, \dots, x_d]]$).

7. Let $(R, \mathfrak{m})$ be a strongly F-regular local ring of characteristic $p > 0$.
   (a) Let $(A, \mathfrak{m}_A, K) \hookrightarrow (B, \mathfrak{m}_B, L)$ be a flat local homomorphism between F-finite reduced local rings such that $\mathfrak{m}_A B = \mathfrak{m}_B$ and L/K is a separable algebraic extension. Show that $\mathrm{fpt}_A(\mathfrak{a}) = \mathrm{fpt}_B(\mathfrak{a}B)$ for any nonzero ideal $\mathfrak{a}$ in A (Hint: use the fact that the natural map $A^{1/p^e} \otimes_A B \to B^{1/p^e}$ is an isomorphism for each $e \geq 1$ under the above coniditons).
   (b) Show that if $\dim R = 1$, then R is regular (Hint: use (a) to reduce to the case where the residue field of R is infinite).
   (c) Conclude that R is normal.

8. Compute the F-pure threshold $\mathrm{fpt}(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ of the local ring R of characteristic $p > 0$.
   (a) $(R, \mathfrak{m}) = k[[x, y, z]]/(xy + z^2)$
   (b) $(R, \mathfrak{m}) = k[[x^3, x^2y, xy^2, y^3]] \subset k[[x, y]]$
   (c) $(R, \mathfrak{m}) = k[[x, y, z]]/(x^2 + y^3 + z^5)$ $(p > 5)$

Day 3

Let $A = k[[x_1, \ldots, x_n]]$ be a formal power series ring over a field k. A *local monomial order* on A is a total order $>$ on the set of all monomials in A satisfying the following two conditions:
(i) $1 > x_i$ for every $i = 1, \ldots, n$, and
(ii) if $u > v$ and w is a monomial, then $uw > vw$.
One of the simplest examples of a local monomial order on A is degree-anticompatible lexicographic order (alex for short), which first sorts by total degree, lower degree terms preceding higher degree terms, and which sorts monomials of the same total degree lexicographically. For example, in $k[[x, y]]$, we have

$$1 >_{\text{alex}} x >_{\text{alex}} y >_{\text{alex}} x^2 >_{\text{alex}} xy >_{\text{alex}} y^2 >_{\text{alex}} x^3 >_{\text{alex}} \cdots.$$

9. Let $R = k[[x, y]]$ be the 2-dimensional formal power series ring over a perfect field k of characteristic $p > 0$.
   (a) Let $\mathfrak{a}$ be a monomial ideal of R. Show that
   $$\text{fpt}(\mathfrak{a}) = \max\{t \in \mathbb{R}_{\geq 0} \mid (1, 1) \in P(t \cdot \mathfrak{a})\}.$$
   (b) Fix a local monomial order $>$ on R. Given a proper ideal $\mathfrak{b}$ of R, let $\text{in}_>(\mathfrak{b})$ be the initial ideal of $\mathfrak{b}$ with respect to $>$. Show that
   $$\text{fpt}(\mathfrak{b}) \geq \text{fpt}(\text{in}_>(\mathfrak{b})) = \max\{t \in \mathbb{R}_{\geq 0} \mid (1, 1) \in P(t \cdot \text{in}_>(\mathfrak{b}))\}.$$
   (c) Given a polynomial $f = \sum_{i,j} c_{ij} x^{a_i} y^{b_j} \in k[x, y]$, let
   $$\mathfrak{a}_f = (x^{a_i} y^{b_j} \mid c_{ij} \neq 0) \subset R$$
   denote the ideal in R generated by monomials appearing in f with nonzero coefficient. When $f = (x + y)^2 - (x - y)^3$, verify that
   $$\text{fpt}(\mathfrak{a}_f) \gneqq \text{fpt}(f).$$

10. Compute the log canonical threshold $\text{lct}_0(f)$ at the origin.
   (a) $f = x^4 + xy^2 + y^5 \in \mathbb{C}[x, y]$
   (b) $f = x^4 + x^2y + xy^2 + y^4 \in \mathbb{C}[x, y]$
   (c) $f = x^3 + y^3 + z^3 + \lambda xyz \in \mathbb{C}[x, y, z] \quad (|\lambda| \neq 3)$

11. Let $f = \sum_{i=1}^{r} c_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \in \mathbb{Z}[x_1, \ldots, x_n]$, where the c_i are nonzero integers and the α_{ij} are nonnegative integers such that for every $1 \leq j \leq n$ there exists $1 \leq i \leq r$ with $\alpha_{ij} > 0$. Set $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in}) \in \mathbb{Z}_{\geq 0}^n$ and suppose that $\alpha_1, \ldots, \alpha_r$ are affinely independent.
   (a) Let Q be the bounded convex polytope defined as
   $$\left\{ (x_1, \ldots, x_r) \in \mathbb{R}_{\geq 0}^r \;\middle|\; \sum_{i=1}^{r} x_i \alpha_{ij} \leq 1 \text{ for all } 1 \leq j \leq n \right\}.$$

Then show that

$$\operatorname{lct}_0(f) = \min\left\{1, \max_{(x_1,\dots,x_r)\in Q} \sum_{i=1}^{r} x_i\right\}$$

(Hint: use the fact that f has non-degenerate principal part).

(b) Let $v = (v_1, \dots, v_r)$ be any element of $Q \cap \mathbb{Q}^r_{\geq 0}$ such that $\sum_{i=1}^{r} v_i = \operatorname{lct}_0(f)$. Take an integer $N \geq 1$ such that $Nv_i \in \mathbb{Z}$ for all $1 \leq i \leq r$. Then show that if $p \equiv 1 \bmod N$, then

$$\nu_e(f_p) \geq \sum_{i=1}^{r} (p^e - 1)v_i = \operatorname{lct}_0(f)(p^e - 1)$$

for all $e \geq 1$. In particular, $\operatorname{fpt}(f_p) = \operatorname{lct}_0(f)$ if $p \equiv 1 \bmod N$.

Day 4

12. Given a rational number $\lambda \in (0,1]$, let $\lambda = \sum_{e\geq 1} \lambda_e/p^e$ denote the non-terminating base p expansion of λ, that is, $0 \leq \lambda_e \leq p-1$ for each $e \in \mathbb{Z}_{\geq 1}$ and there exist infinitely many e with $\lambda_e \neq 0$. For example, the non-terminating base p expansion of $1/p$ is $\sum_{e\geq 2}(p-1)/p^e$. For every $e \in \mathbb{Z}_{\geq 1}$, let
$$\langle \lambda \rangle_e = \sum_{k=1}^{e} \frac{\lambda_k}{p^k}.$$
Let $f \in \mathfrak{m}$ be a non-unit of an F-finite F-pure local domain $(R, \mathfrak{m})$ of characteristic $p > 0$. Fix any integer $e \geq 1$.
    (a) Show that there exists an integer $\ell \geq 1$ such that $\nu_\ell(f) \geq \lceil p^\ell \langle \mathrm{fpt}(f) \rangle_e \rceil$.
    (b) Show that $\nu_e(f) \geq p^e \langle \mathrm{fpt}(f) \rangle_e$.
    (c) Show that $p^e \langle \mathrm{fpt}(f) \rangle_e + 1 = \lceil p^e \mathrm{fpt}(f) \rceil$.
    (d) Conclude that $\nu_e(f) = p^e \langle \mathrm{fpt}(f) \rangle_e$.

13. Compute the Bernstein-Sato polynomial $b_f(s)$ and the local Bernstein-Sato polynomial $b_{f,0}(s)$.
    (a) $f = x(x+y+1) \in \mathbb{C}[x,y]$
    (b) $f = x^3 \in \mathbb{C}[x,y]$

14. Let $f = x^2 + y^5 \in \mathbb{Z}[x,y]$.
    (a) For every prime number p, compute the F-pure threshold $\mathrm{fpt}(f_p)$.
    (b) It is known that $b_{f,0}(s)$ has 5 distinct roots. Find all of them using (a).

15. Let $R = \bigoplus_{n\geq 0} R_n = k[x,y,z]$ be the 3-dimensional polynomial ring over k with standard grading and $f \in R_3$ be a homogeneous polynomial of degree 3 such that $\sqrt{\mathrm{Jac}(f)} = (x,y,z)$. Let $f_x = \partial f/\partial x$ (resp. $f_y = \partial f/\partial y$, $f_z = \partial f/\partial z$).
    (a) Verify that f_x, f_y, f_z form a homogeneous regular sequence.
    (b) Given a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, let
    $$H_M(t) = \sum_{n\in\mathbb{Z}} \ell(M_n)t^n \in \mathbb{Z}[[t]].$$
    Show that $H_{R(i)}(t) = t^{-i}/(1-t)^3$ for every $i \in \mathbb{Z}$, where $R(i)$ is the graded R-module defined as $[R(i)]_j = R_{i+j}$ for each $j \in \mathbb{Z}$.
    (c) Show that $(1-t^2)H_R(t) = H_{R/(f_x)}(t)$ (Hint: use the fact that if $0 \to L \to M \to N \to 0$ is an exact sequence of graded R-modules, then $H_M(t) = H_L(t) + H_N(t)$).
    (d) Show that $H_{R/\mathrm{Jac}(f)}(t) = (1+t^3)$.
    (e) Conclude that $\mathfrak{m}^4 \subset \mathrm{Jac}(f)$, where $\mathfrak{m} = \bigoplus_{n\geq 1} R_n$ is the unique homogeneous maximal ideal of R.