THE CATEGORY OF MAXIMAL COHEN–MACAULAY MODULES

RYO TAKAHASHI

Contents

Overview

A Cohen–Macaulay local ring R is said to be of *finite representation type* if

$$\#\{\text{Indecomposable maximal Cohen–Macaulay } R\text{-modules}\}/_{\cong} < \infty$$

Theorem (Auslander–Huneke–Leuschke–Wiegand)**.** A Cohen–Macaulay local ring R of finite representation type has an isolated singularity

- Auslander (1986): R complete
- Leuschke–Wiegand (2000): R excellent
- Huneke–Leuschke (2002): General

Example.
(1) $R = \mathbb{C}[[x, y]]/(xy) \Rightarrow \#\{R, R/(x), R/(y)\} = 3 < \infty$, R an isolated singularity
(2) $R = \mathbb{C}[[x, y]]/(x^2) \Rightarrow R$ not an isolated singularity, $\#\{R, (x), (x, y), (x, y^2), \dots\} = \infty$

What to do. Analyze the structure of modules over a local ring. More precisely:

(1) Study generation in the module category using annihilators of Tor, Ext
(2) Refine the Auslander–Huneke–Leuschke–Wiegand theorem
(3) Introduce/investigate the dimension of a subcategory of the module category

Throughout.

- $(R, \mathfrak{m}, k)$ = a noetherian local ring of dimension d
- CM = Cohen–Macaulay
- module = f.g. module
  CM module = maximal CM module

- $\operatorname{mod} R = \{R\text{-modules}\} \supseteq \begin{cases} \mathrm{CM}(R) = \{\mathrm{CM}\ R\text{-modules}\} \\ \mathrm{fl}\, R = \{R\text{-modules of finite length}\} \end{cases}$

1. Uniform annihilation of Tor, Ext

Lemma 1.1. Let $a \in R$, $n \in \mathbb{Z}$, $t \in \mathbb{N}$, $M \in \operatorname{mod} R$

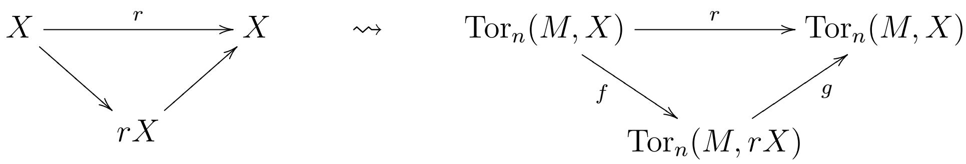
(1) Suppose
$$a \operatorname{Tor}_n^R(M,X) = a \operatorname{Tor}_{n-1}^R(M,X) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t$. Then
$$a^2 \operatorname{Tor}_n^R(M,X) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t+1$

(2) (a) Suppose
$$a \operatorname{Ext}_R^n(M,X) = a \operatorname{Ext}_R^{n+1}(M,X) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t$. Then
$$a^2 \operatorname{Ext}_R^n(M,X) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t+1$

(b) Suppose
$$a \operatorname{Ext}_R^n(X,M) = a \operatorname{Ext}_R^{n+1}(X,M) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t$. Then
$$a^2 \operatorname{Ext}_R^n(X,M) = 0$$
for all $X \in \operatorname{mod} R$ with $\dim X \leqslant t+1$

Proof. (1) Fix $X \in \operatorname{mod} R$ with $\dim X \leqslant t+1$. May assume $\dim X = t+1$

Take $r \in R$ a ssop of X
$$\begin{cases} 0 \to (0 :_X r) \to X \to rX \to 0 \\ 0 \to rX \to X \to X/rX \to 0 \end{cases}$$
$$\rightsquigarrow \begin{cases} \operatorname{Tor}_n(M,X) \xrightarrow{f} \operatorname{Tor}_n(M,rX) \to \operatorname{Tor}_{n-1}(M,(0 :_X r)) \\ \operatorname{Tor}_n(M,rX) \xrightarrow{g} \operatorname{Tor}_n(M,X) \to \operatorname{Tor}_n(M,X/rX) \end{cases}$$
Since $\dim X/rX = t$ and $\dim(0 :_X r) \leqslant t$ by Exercise 1(2),
$$a \operatorname{Tor}_n(M,X/rX) = a \operatorname{Tor}_{n-1}(M,(0 :_X r)) = 0$$
$$\begin{array}{ccccc} \operatorname{Tor}_n(M,X) & \xrightarrow{f} & \operatorname{Tor}_n(M,rX) & \longrightarrow & \operatorname{Tor}_{n-1}(M,(0 :_X r)) \\ {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow{\scriptstyle 0} \\ \operatorname{Tor}_n(M,X) & \xrightarrow{f} & \operatorname{Tor}_n(M,rX) & \longrightarrow & \operatorname{Tor}_{n-1}(M,(0 :_X r)) \end{array}$$
$$\begin{array}{ccccc} \operatorname{Tor}_n(M,rX) & \xrightarrow{g} & \operatorname{Tor}_n(M,X) & \longrightarrow & \operatorname{Tor}_n(M,X/rX) \\ {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow{\scriptstyle 0} \\ \operatorname{Tor}_n(M,rX) & \xrightarrow{g} & \operatorname{Tor}_n(M,X) & \longrightarrow & \operatorname{Tor}_n(M,X/rX) \end{array}$$



Hence:

$$\begin{aligned} y \in \operatorname{Tor}_n(M,X) &\Rightarrow \exists\, z \in \operatorname{Tor}_n(M,rX) \text{ s.t. } ay = g(z) \\ &\Rightarrow \exists\, w \in \operatorname{Tor}_n(M,X) \text{ s.t. } az = f(w) \\ &\Rightarrow a^2 y = ag(z) = g(az) = gf(w) = rw \end{aligned}$$

Thus:

$$a^2 \operatorname{Tor}_n(M,X) \subseteq r \operatorname{Tor}_n(M,X)$$

This holds for all ssop $r \in R$ of X ⇝ Can replace r with r^j $\forall\, j > 0$

$$\therefore\ a^2 \operatorname{Tor}_n(M,X) \subseteq \bigcap_{j>0} r^j \operatorname{Tor}_n(M,X) = 0$$

by Krull's intersection theorem

(2) Similar to (1). (Try!) ■

Proposition 1.2. Let $a \in R$, $n \in \mathbb{Z}$

(1) Suppose

$$a \operatorname{Tor}_i^R(M,N) = 0$$

for all $n - 2d \leqslant i \leqslant n$ and $M, N \in \operatorname{fl} R$. Then

$$a^{2^{2d}} \operatorname{Tor}_n^R(M,N) = 0$$

for all $M, N \in \operatorname{mod} R$

(2) Suppose

$$a \operatorname{Ext}_R^i(M,N) = 0$$

for all $n \leqslant i \leqslant n + 2d$ and $M, N \in \operatorname{fl} R$. Then

$$a^{2^{2d}} \operatorname{Ext}_R^n(M,N) = 0$$

for all $M, N \in \operatorname{mod} R$

Proof. (1) Fix $M \in \operatorname{fl} R$. By assumption and Lemma 1.1(1):

$$a \operatorname{Tor}_i(M,N) = 0 \text{ for all} \begin{cases} n - 2d \leqslant i \leqslant n \\ N \in \operatorname{fl} R \end{cases}$$

$$a^2 \operatorname{Tor}_i(M,N) = 0 \text{ for all} \begin{cases} n - 2d + 1 \leqslant i \leqslant n \\ N \in \operatorname{mod} R \text{ with } \dim N \leqslant 1 \end{cases}$$

$$a^4 \operatorname{Tor}_i(M,N) = 0 \text{ for all} \begin{cases} n - 2d + 2 \leqslant i \leqslant n \\ N \in \operatorname{mod} R \text{ with } \dim N \leqslant 2 \end{cases}$$

$$\cdots$$

and get

$$a^{2^d}\operatorname{Tor}_i(M,N)=0 \text{ for all } \begin{cases} n-d\leqslant i\leqslant n \\ M\in \operatorname{fl}R,\ N\in \operatorname{mod}R\end{cases}$$

Next, fix $N \in \operatorname{mod} R$. Similarly:

$$a^{2^d}\operatorname{Tor}_i(M,N)=0 \text{ for all } \begin{cases} n-d\leqslant i\leqslant n \\ M\in \operatorname{fl}R\end{cases}$$

$$a^{2^{d+1}}\operatorname{Tor}_i(M,N)=0 \text{ for all } \begin{cases} n-d+1\leqslant i\leqslant n \\ M\in \operatorname{mod}R \text{ with } \dim M\leqslant 1\end{cases}$$

$$a^{2^{d+2}}\operatorname{Tor}_i(M,N)=0 \text{ for all } \begin{cases} n-d+2\leqslant i\leqslant n \\ M\in \operatorname{mod}R \text{ with } \dim M\leqslant 2\end{cases}$$

$$\cdots$$

and get

$$a^{2^{2d}}\operatorname{Tor}_n(M,N)=0 \text{ for all } M,N\in\operatorname{mod}R$$

(2) Similar to (1) ■

Notation 1.3. For $\mathcal{X},\mathcal{Y}\subseteq \operatorname{mod}R$,

$$\operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y}) := \bigcap_{i>0,\,X\in\mathcal{X},\,Y\in\mathcal{Y}} \operatorname{ann}_R \operatorname{Tor}_i^R(X,Y)$$

$$\operatorname{ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y}) := \bigcap_{i>0,\,X\in\mathcal{X},\,Y\in\mathcal{Y}} \operatorname{ann}_R \operatorname{Ext}_R^i(X,Y)$$

When $\mathcal{X}=\mathcal{Y}$, simply write $\operatorname{ann}\operatorname{Tor}\mathcal{X}$, $\operatorname{ann}\operatorname{Ext}\mathcal{X}$

Definition 1.4. (1) The *punctured spectrum* of R is

$$\operatorname{Spec}_0 R := \operatorname{Spec} R\setminus\{\mathfrak{m}\}$$

(2) When R is CM,

$$\operatorname{CM}_0(R) := \{M\in\operatorname{CM}(R)\mid M_{\mathfrak{p}} \text{ is } R_{\mathfrak{p}}\text{-free for all } \mathfrak{p}\in\operatorname{Spec}_0 R\}$$

(3) Let

$$\cdots\to F_n \xrightarrow{\partial_n} F_{n-1}\to\cdots\to F_1\xrightarrow{\partial_1} F_0\xrightarrow{\partial_0} M\to 0$$

be the *minimal* free resolution of $M\in\operatorname{mod}R$. The *nth syzygy* of M is

$$\Omega^n M := \operatorname{Im}\partial_n$$

Example 1.5.
(1) $R=\mathbb{C}[[x,y]]/(xy) \Rightarrow \operatorname{ann}\operatorname{Tor}\operatorname{CM}_0(R)=\operatorname{ann}\operatorname{Ext}\operatorname{CM}_0(R)=(x,y)=\mathfrak{m}$
(2) $R=\mathbb{C}[[x,y]]/(x^2) \Rightarrow \operatorname{ann}\operatorname{Tor}\operatorname{CM}_0(R)=\operatorname{ann}\operatorname{Ext}\operatorname{CM}_0(R)=(x)$

Proposition 1.6. Let R be CM
(1) $a\in\operatorname{ann}\operatorname{Tor}\operatorname{CM}_0(R) \implies a^{2^{2d}}\operatorname{Tor}_i^R(M,N)=0\ \ \forall\, i>4d,\ M,N\in\operatorname{mod}R$
(2) $a\in\operatorname{ann}\operatorname{Ext}\operatorname{CM}_0(R) \implies a^{2^{2d}(d+1)}\operatorname{Ext}_R^i(M,N)=0\ \ \forall\, i>d,\ M,N\in\operatorname{mod}R$

Proof. Let $M, N \in \operatorname{fl} R$. Then $\Omega^d M, \Omega^d N \in \mathrm{CM}_0(R)$ by Exercise 3(3b)

(1) As $a \operatorname{Tor}_{>0}(\Omega^d M, \Omega^d N) = 0$, we have $a \operatorname{Tor}_{>2d}(M, N) = 0$

$$\begin{aligned} n > 4d \ &\Rightarrow\ a \operatorname{Tor}_i^R(M, N) = 0 \quad n - 2d \leqslant \forall\, i \leqslant n \\ &\Rightarrow\ a^{2^{2d}} \operatorname{Tor}_n^R(X, Y) = 0 \quad \forall\, X, Y \in \operatorname{mod} R \end{aligned}$$

by Proposition 1.2(1)

(2) Fix $i > 0$

$$\begin{aligned} &\begin{cases} a \operatorname{Ext}_R^i(K, L) = 0 \ \ \forall\, K, L \in \mathrm{CM}_0(R) \\ \forall\, j \geqslant 0,\ \exists\, 0 \to \Omega^{j+1} N \to F_j \to \Omega^j N \to 0,\ F_j \text{ free} \end{cases} \\ \rightsquigarrow &\begin{cases} a \operatorname{Ext}_R^i(K, F_j) = a \operatorname{Ext}_R^i(K, \Omega^d N) = 0 \\ \operatorname{Ext}_R^i(K, F_j) \to \operatorname{Ext}_R^i(K, \Omega^j N) \to \operatorname{Ext}_R^{i+1}(K, \Omega^{j+1} N) \end{cases} \\ \rightsquigarrow \ \ & a^{d+1} \operatorname{Ext}_R^i(K, N) = 0 \end{aligned}$$

by an inductive argument using Exercise 6
Letting $K := \Omega^d M$, we have $a^{d+1} \operatorname{Ext}_R^{>d}(M, N) = 0$

$$\therefore\ (a^{d+1})^{2^{2d}} \operatorname{Ext}_R^h(X, Y) = 0 \quad \forall\, h > d,\ X, Y \in \operatorname{mod} R$$

by Proposition 1.2(2) ■

Notation 1.7. The *singular locus* of R is

$$\operatorname{Sing} R := \{\mathfrak{p} \in \operatorname{Spec} R \mid R_\mathfrak{p} \text{ is not regular}\}$$

Proposition 1.8. Let R be CM. Then

$$\operatorname{Sing} R \subseteq \mathrm{V}(\operatorname{ann} \operatorname{Tor} \mathrm{CM}_0(R)) \cap \mathrm{V}(\operatorname{ann} \operatorname{Ext} \mathrm{CM}_0(R))$$

Proof. Let $\mathfrak{p} \in \operatorname{Sing} R$. Take $a \in \operatorname{ann} \operatorname{Tor} \mathrm{CM}_0(R)$. By Proposition 1.6(1),

$$a^{2^{2d}} \operatorname{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{p}) = 0 \quad \forall\, i > 4d$$

Localization at $\mathfrak{p}$ gives

$$a^{2^{2d}} \operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p})) = 0 \quad \forall\, i > 4d$$

We have

$$\begin{aligned} a \notin \mathfrak{p} \ &\Longrightarrow\ a^{2^{2d}} \text{ is a unit in } R_\mathfrak{p} \\ &\Longrightarrow\ \operatorname{Tor}_i^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \kappa(\mathfrak{p})) = 0 \text{ for all } i > 4d \\ &\Longrightarrow\ R_\mathfrak{p} \text{ is regular} \quad (\times) \end{aligned}$$

Hence $a \in \mathfrak{p}$. Thus $\operatorname{Sing} R \subseteq \mathrm{V}(\operatorname{ann} \operatorname{Tor} \mathrm{CM}_0(R))$.

The assertion for Ext is similarly shown ■

2. Annihilators and nonfree loci

Definition 2.1. (1) Let

$$\cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

be the minimal free resolution of $M \in \operatorname{mod} R$. Set $(-)^* = \operatorname{Hom}_R(-, R)$. The *transpose* of M is

$$\operatorname{Tr} M := \operatorname{Coker}(\partial_1^*)$$

Thus

$$\exists\, 0 \to M^* \to F_0^* \xrightarrow{\partial_1^*} F_1^* \to \operatorname{Tr} M \to 0$$

(2) For $M, N \in \operatorname{mod} R$,

$$\underline{\operatorname{Hom}}_R(M,N) := \operatorname{Hom}_R(M,N)/\operatorname{P}_R(M,N),$$

where $\operatorname{P}_R(M,N) = \left\{ f \;\middle|\; \begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle \exists g}\searrow \quad \nearrow{\scriptstyle \exists h} & \\ & R^{\oplus} & \end{array} \right\}$

Lemma 2.2. For $M, N \in \operatorname{mod} R$,

$$\underline{\operatorname{Hom}}_R(M,N) \cong \operatorname{Tor}_1^R(\operatorname{Tr} M, N)$$

Proof. Exercise 8 ■

Proposition 2.3. For $M \in \operatorname{mod} R$,

$$\begin{aligned} \operatorname{ann} \operatorname{Ext}(M, \operatorname{mod} R) &= \operatorname{ann}_R \operatorname{Ext}_R^1(M, \Omega M) \\ &= \operatorname{ann}_R \operatorname{Tor}_1^R(M, \operatorname{Tr} M) = \operatorname{ann} \operatorname{Tor}(M, \operatorname{mod} R) \end{aligned}$$

Proof. Clearly,

$$\begin{aligned} \operatorname{ann} \operatorname{Ext}(M, \operatorname{mod} R) &\subseteq \operatorname{ann}_R \operatorname{Ext}_R^1(M, \Omega M) \\ \operatorname{ann} \operatorname{Tor}(M, \operatorname{mod} R) &\subseteq \operatorname{ann}_R \operatorname{Tor}_1^R(M, \operatorname{Tr} M) \end{aligned}$$

Enough to show:

$$\operatorname{ann} \operatorname{Ext}^1(M, \Omega M) \cup \operatorname{ann} \operatorname{Tor}_1(M, \operatorname{Tr} M) \subseteq \operatorname{ann} \operatorname{Tor}(M, \operatorname{mod} R) \cap \operatorname{ann} \operatorname{Ext}(M, \operatorname{mod} R)$$

(1) Let $a \in \operatorname{ann}_R \operatorname{Ext}_R^1(M, \Omega M)$

$$\exists\, 0 \to \Omega M \to F \xrightarrow{\pi} M \to 0, \ F \text{ free}$$

$$\rightsquigarrow \quad \begin{array}{ccccc} \operatorname{Hom}_R(M,F) & \xrightarrow{\operatorname{Hom}_R(M,\pi)} & \operatorname{Hom}_R(M,M) & \longrightarrow & \operatorname{Ext}_R^1(M,\Omega M) \\ {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow{\scriptstyle 0} \\ \operatorname{Hom}_R(M,F) & \xrightarrow{\operatorname{Hom}_R(M,\pi)} & \operatorname{Hom}_R(M,M) & \longrightarrow & \operatorname{Ext}_R^1(M,\Omega M) \end{array}$$

$$\operatorname{id}_M \in \operatorname{Hom}_R(M,M) \ \rightsquigarrow \ (M \xrightarrow{a} M) = \operatorname{Hom}_R(M,\pi)(f) = \pi f \quad \exists\, f \in \operatorname{Hom}_R(M,F)$$

$$\begin{array}{ccc} M & \xrightarrow{\quad a \quad} & M \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & F & \end{array}$$

For all $i > 0$ and $N \in \operatorname{mod} R$:

$$\begin{array}{ccccccc} \operatorname{Tor}_i(M,N) & \xrightarrow{a} & \operatorname{Tor}_i(M,N) & \qquad & \operatorname{Ext}^i(M,N) & \xrightarrow{a} & \operatorname{Ext}^i(M,N) \\ & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\ & \operatorname{Tor}_i(F,N) & & & & \operatorname{Ext}^i(F,N) & \end{array}$$

Hence $a \in \operatorname{ann}\operatorname{Tor}(M, \operatorname{mod} R) \cap \operatorname{ann}\operatorname{Ext}(M, \operatorname{mod} R)$

(2) Let $b \in \operatorname{ann}_R \operatorname{Tor}_1^R(M, \operatorname{Tr} M)$. By Lemma 2.2,

$$0 = b\,\underline{\operatorname{Hom}}_R(M,M) \ni b \cdot \underline{\operatorname{id}_M} = \underline{(M \xrightarrow{b} M)}$$

Hence $(M \xrightarrow{b} M)$ factors through a free module

Similarly to (1), $b \in \operatorname{ann}\operatorname{Tor}(M, \operatorname{mod} R) \cap \operatorname{ann}\operatorname{Ext}(M, \operatorname{mod} R)$ ■

Definition 2.4. The *nonfree locus* of $M \in \operatorname{mod} R$ is

$$\operatorname{NF}(M) := \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \text{ is not } R_\mathfrak{p}\text{-free}\}$$

Lemma 2.5. Let $M \in \operatorname{mod} R$. Then

$$\operatorname{NF}(M) = \operatorname{Supp}\operatorname{Ext}_R^1(M, \Omega M)$$

In particular, $\operatorname{NF}(M)$ is a closed subset of $\operatorname{Spec} R$

Proof. Exercise 9 ■

Proposition 2.6. For $M \in \operatorname{mod} R$,

$$\operatorname{NF}(M) = \operatorname{V}(\operatorname{ann}\operatorname{Tor}(M, \operatorname{mod} R)) = \operatorname{V}(\operatorname{ann}\operatorname{Ext}(M, \operatorname{mod} R))$$

Proof. Combine Lemma 2.5 and Proposition 2.3 ■

3. Finite generation in the module category

Notation 3.1. Let $\mathcal{X}, \mathcal{Y} \subseteq \operatorname{mod} R$ and $r \geqslant 1$

(1) $\operatorname{add}\mathcal{X} = \{$Direct summands of finite direct sums of modules in $\mathcal{X}\}$

(2) $[\mathcal{X}] = \operatorname{add}\{R, \Omega^i X \mid i \geqslant 0,\ X \in \mathcal{X}\}$

(3) $\begin{cases} \mathcal{X} \circ \mathcal{Y} = \{M \in \operatorname{mod} R \mid \exists\, 0 \to X \to M \to Y \to 0 \text{ with } X \in \mathcal{X},\ Y \in \mathcal{Y}\}, \\ \mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]] \end{cases}$

(4) $[\mathcal{X}]_r = \begin{cases} [\mathcal{X}] & (r = 1) \\ [\mathcal{X}]_{r-1} \bullet \mathcal{X} = [[\mathcal{X}]_{r-1} \circ [\mathcal{X}]] & (r \geqslant 2) \end{cases}$

Remark 3.2. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \operatorname{mod} R$

(1) $M \in \mathcal{X} \bullet \mathcal{Y} \iff \exists\, 0 \to X \to Z \to Y \to 0$ with $X \in [\mathcal{X}]$, $Y \in [\mathcal{Y}]$ s.t. $M \lessdot Z$

(2) $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z} = \mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$, $\quad [\mathcal{X}]_a \bullet [\mathcal{X}]_b = [\mathcal{X}]_{a+b}\ \forall\, a, b > 0$

Lemma 3.3. Let $\mathcal{X}, \mathcal{Y} \subseteq \operatorname{mod} R$ and $n \geqslant 0$

$$(\operatorname{ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y}))^n \subseteq \operatorname{ann}\operatorname{Tor}([\mathcal{X}]_n, \mathcal{Y}) \subseteq \operatorname{ann}\operatorname{Tor}(\mathcal{X}, \mathcal{Y})$$
$$(\operatorname{ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y}))^n \subseteq \operatorname{ann}\operatorname{Ext}([\mathcal{X}]_n, \mathcal{Y}) \subseteq \operatorname{ann}\operatorname{Ext}(\mathcal{X}, \mathcal{Y})$$

$$\therefore \begin{cases} \mathrm{V}(\operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})) = \mathrm{V}(\operatorname{ann}\operatorname{Tor}([\mathcal{X}]_n,\mathcal{Y})) \\ \mathrm{V}(\operatorname{ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y})) = \mathrm{V}(\operatorname{ann}\operatorname{Ext}([\mathcal{X}]_n,\mathcal{Y})) \end{cases}$$

Proof. Only show:

$$(\operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y}))^n \subseteq \operatorname{ann}\operatorname{Tor}([\mathcal{X}]_n,\mathcal{Y}) \subseteq \operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})$$

The other is similarly shown

2nd ($\subseteq$): OK as $\mathcal{X} \subseteq [\mathcal{X}]_n$
1st ($\subseteq$): Enough to show

$$\operatorname{ann}\operatorname{Tor}([\mathcal{X}]_n,\mathcal{Y}) \begin{cases} = \operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y}) & (n=1) \\ \supseteq \operatorname{ann}\operatorname{Tor}([\mathcal{X}]_{n-1},\mathcal{Y}) \cdot \operatorname{ann}\operatorname{Tor}([\mathcal{X}],\mathcal{Y}) & (n \geqslant 2) \end{cases}$$

Take $a \in \operatorname{ann}\operatorname{Tor}([\mathcal{X}]_{n-1},\mathcal{Y})$, $b \in \operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})$
Fix $i > 0$, $Z \in [\mathcal{X}]_n$, $Y \in \mathcal{Y}$

$$\exists\, 0 \to L \to M \to N \to 0 \text{ with } L \in [\mathcal{X}]_{n-1},\ N \in [\mathcal{X}],\ Z \lessdot M$$
$$\rightsquigarrow \operatorname{Tor}_i(L,Y) \to \operatorname{Tor}_i(M,Y) \to \operatorname{Tor}_i(N,Y)$$
$$\rightsquigarrow a \cdot b \in \operatorname{ann}\operatorname{Tor}_i(L,Y) \cdot \operatorname{Tor}_i(N,Y) \subseteq \operatorname{ann}\operatorname{Tor}_i(M,Y) \subseteq \operatorname{ann}\operatorname{Tor}_i(Z,Y)$$

by Exercise 6 ■

Definition 3.4. $\mathcal{X} \subseteq \operatorname{mod} R$ is *finitely generated* if $[G]_n = \mathcal{X}$ for some $G \in \operatorname{mod} R$ and $n > 0$

Proposition 3.5. $\begin{cases} \mathcal{X} \subseteq \mathrm{CM}_0(R) \text{ f.g.} \\ \mathcal{Y} \subseteq \operatorname{mod} R \end{cases} \Longrightarrow \begin{cases} \operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y}) \\ \operatorname{ann}\operatorname{Ext}(\mathcal{X},\mathcal{Y}) \end{cases}$ are $\mathfrak{m}$-primary or unit

Proof. $\exists\, G \in \operatorname{mod} R$, $\exists\, n > 0$ s.t. $[G]_n = \mathcal{X}$

$$\mathrm{V}(\operatorname{ann}\operatorname{Tor}(\mathcal{X},\mathcal{Y})) \overset{\text{(a)}}{=} \mathrm{V}(\operatorname{ann}\operatorname{Tor}(G,\mathcal{Y})) \subseteq \mathrm{V}(\operatorname{ann}\operatorname{Tor}(G,\operatorname{mod} R)) \overset{\text{(b)}}{=} \mathrm{NF}(G) \overset{\text{(c)}}{\subseteq} \{\mathfrak{m}\}$$

where $\begin{cases} \text{(a)} \Leftarrow \text{Lemma 3.3} \\ \text{(b)} \Leftarrow \text{Proposition 2.6} \\ \text{(c)} \Leftarrow G \in \mathrm{CM}_0(R) \end{cases}$ The assertion for Ext is shown similarly ■

Definition 3.6. R has an *isolated singularity* if $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}_0 R$, or in other words, if $\operatorname{Sing} R \subseteq \{\mathfrak{m}\}$

Theorem 3.7 (Dao–T (2015))**.** Let $(R,\mathfrak{m})$ be a CM local ring. Consider:
(a) $\mathrm{CM}_0(R)$ is finitely generated
(b) $\operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R)$ is $\mathfrak{m}$-primary or unit
(c) $\operatorname{ann}\operatorname{Tor}\mathrm{CM}_0(R)$ is $\mathfrak{m}$-primary or unit
(d) R has an isolated singularity

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d)

Proof. Combine Propositions 3.5 and 1.8 ■

Notation 3.8. For $\mathcal{X} \subseteq \operatorname{mod} R$,

$$\operatorname{ind}\mathcal{X} := \{M \in \mathcal{X} \mid M \text{ is indecomposable}\}/_{\cong}$$

Corollary 3.9 (Refinement of Auslander–Huneke–Leuschke–Wiegand)**.**
Let R be a CM local ring. Suppose

$$\#\operatorname{ind}\mathrm{CM}_0(R) < \infty$$

Then R has an isolated singularity. Hence R has finite representation type.

Proof. Let

$$\operatorname{ind}\mathrm{CM}_0(R) =: \{M_1, \dots, M_n\}$$

By Exercise 12,

$$M := M_1 \oplus \cdots \oplus M_n \in \mathrm{CM}_0(R), \qquad [M] \subseteq \mathrm{CM}_0(R)$$

Take $N \in \mathrm{CM}_0(R)$. Let

$$N = N_1 \oplus \cdots \oplus N_t \quad \text{an indecomposable decomposition}$$

Then

$$N_i \in \mathrm{CM}_0(R)\ \forall\, i \quad \rightsquigarrow \quad \forall\, i\ \exists\, j \text{ s.t } N_i \cong M_j \quad \rightsquigarrow \quad N \in \operatorname{add} M$$

Therefore

$$\mathrm{CM}_0(R) = \operatorname{add} M = [M] = [M]_1$$

In particular, $\mathrm{CM}_0(R)$ is finitely generated
By Theorem 3.7, R has an isolated singularity
Thus $\mathrm{CM}(R) = \mathrm{CM}_0(R)$ by Exercise 13, and R has finite representation type ∎

4. Improving Theorem 3.7

Lemma 4.1 (T (2014))**.**

$$\text{Let } \begin{cases} M \in \operatorname{mod} R \\ \boldsymbol{x} = x_1, \dots, x_n \subseteq \operatorname{ann}\operatorname{Ext}^1(M, \Omega M) \\ F \text{ a minimal free resolution of } M \end{cases}$$

Then

$$\mathrm{K}(\boldsymbol{x}, M) \cong X := (0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0)$$

$$\text{s.t. } X_i = \begin{cases} \bigoplus_{j=0}^{i} F_j^{\oplus\binom{n}{i-j}} & (0 \leqslant i \leqslant n-1) \\ \bigoplus_{j=0}^{n} (\Omega^j M)^{\oplus\binom{n}{j}} & (i = n) \end{cases}$$

Proof. Induction on n
<u>$n = 1$</u>:

$$\begin{array}{rccccccccc} \sigma: & 0 & \longrightarrow & \Omega M & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \| & & \uparrow & & \uparrow{\scriptstyle x_1} & & \\ x_1\sigma: & 0 & \longrightarrow & \Omega M & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

$$x_1 \operatorname{Ext}^1_R(M, \Omega M) = 0 \ \rightsquigarrow\ x_1\sigma \text{ splits} \ \rightsquigarrow\ N \cong \Omega M \oplus M$$

$$\rightsquigarrow 0 \to W \to X \to \mathrm{K}(x_1, M) \to 0, \text{ where } \begin{cases} W = (0 \to \Omega M \xrightarrow{=} \Omega M \to 0) \\ X = (0 \to \Omega M \oplus M \to F_0 \to 0) \end{cases}$$

As W is acyclic, $\mathrm{K}(x_1, M) \cong X$
<u>$n \geqslant 2$</u>: $\mathrm{K}(\boldsymbol{x}, M) = \mathrm{K}(x_1, \dots, x_{n-1}, M) \otimes \mathrm{K}(x_n, R)$

Apply the induction hypothesis to $\mathrm{K}(x_1, \dots, x_{n-1}, M)$
..... ■

Proposition 4.2. Let R be CM

$$\operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R) \text{ is } \mathfrak{m}\text{-primary or unit} \implies \mathrm{CM}_0(R) \text{ is f.g.}$$

Proof. $\exists\, h > 0$ s.t. $\mathfrak{m}^h \subseteq \operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R)$ Choose a sop $\boldsymbol{x} = x_1, \dots, x_d \subseteq \mathfrak{m}^h$

$$\exists\, R/(\boldsymbol{x}) \supsetneq \mathfrak{m}(R/(\boldsymbol{x})) \supsetneq \mathfrak{m}^2(R/(\boldsymbol{x})) \supsetneq \cdots \supsetneq \mathfrak{m}^r(R/(\boldsymbol{x})) = 0$$

Fix $M \in \mathrm{CM}_0(R) \rightsquigarrow \Omega M \in \mathrm{CM}_0(R)$
$\boldsymbol{x} \subseteq \mathfrak{m}^h \subseteq \operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R) \subseteq \operatorname{ann}\operatorname{Ext}^1(M, \Omega M)$ $\mathrm{K}(\boldsymbol{x}, M) \cong M/\boldsymbol{x}M$
By Lemma 4.1:

$$0 \to X_d \to P_{d-1} \to \cdots \to P_0 \to M/\boldsymbol{x}M \to 0 \quad (P_i \text{ free},\ M \lessdot X_d)$$

$\therefore\ X_d \cong \Omega^d(M/\boldsymbol{x}M)$ up to free summands

$$\exists\, M/\boldsymbol{x}M \supsetneq \mathfrak{m}(M/\boldsymbol{x}M) \supsetneq \mathfrak{m}^2(M/\boldsymbol{x}M) \supsetneq \cdots \supsetneq \mathfrak{m}^r(M/\boldsymbol{x}M) = 0$$
$$\rightsquigarrow\ 0 \to \mathfrak{m}^{i+1}(M/\boldsymbol{x}M) \to \mathfrak{m}^i(M/\boldsymbol{x}M) \to k^{\oplus} \to 0$$
$$\rightsquigarrow\ 0 \to \Omega^d(\mathfrak{m}^{i+1}(M/\boldsymbol{x}M)) \to \Omega^d(\mathfrak{m}^i(M/\boldsymbol{x}M)) \oplus R^{\oplus} \to \Omega^d k^{\oplus} \to 0$$
$$\rightsquigarrow\ \Omega^d(M/\boldsymbol{x}M) \in [\Omega^d k]_r \ \rightsquigarrow\ M \in [\Omega^d k]_r$$

Thus $\mathrm{CM}_0(R) = [\Omega^d k]_r$ ■

Definition 4.3. Let R be complete and equicharacteristic $\rightsquigarrow R = \dfrac{k[[x_1, \dots, x_n]]}{(f_1, \dots, f_m)}$
$h := \mathrm{ht}(f_1, \dots, f_m)$

$$\operatorname{jac} R = \mathrm{I}_h\left(\frac{\partial f_i}{\partial x_j}\right) R \quad \text{the } \textit{Jacobian ideal} \text{ of } R$$

Lemma 4.4. R complete, equicharacteristic
(1) (Jacobian criterion) $\operatorname{Sing} R = \mathrm{V}(\operatorname{jac} R)$ if k is perfect
(2) (Wang (1994)) $\operatorname{jac} R \subseteq \operatorname{ann}\operatorname{Ext}(\mathrm{CM}(R), \operatorname{mod} R)$ if R is CM

Proof. (1) See [16, §30], [19, §5] and [20, §6]
(2) Show:

- $\operatorname{jac} R = \sum_{A \subseteq R \text{ Noether normalizations}} \mathrm{J}(R/A)$,
  where $R = A[y_1, \dots, y_t]/(g_1, \dots, g_s)$, $\mathrm{J}(R/A) = \mathrm{I}_t(\partial g_i/\partial y_j)R$
- $\mathrm{J}(R/A) \subseteq \mathrm{N}(R/A)$,
  where $\mu : S = R \otimes_A R \to R$, $a \otimes b \mapsto ab$, $\mathrm{N}(R/A) = \mu(\operatorname{ann}_S \operatorname{Ker} \mu)$
- $\mathrm{N}(R/A) \cdot \operatorname{ann}_A \operatorname{Ext}^1_A(M, N) \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, N)$ for $M, N \in \operatorname{mod} R$

Hence:

$$M \in \mathrm{CM}(R),\ N \in \operatorname{mod} R \ \Rightarrow\ \mathrm{N}(R/A) \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, N)$$
$$\Rightarrow\ \operatorname{jac} R \subseteq \sum \mathrm{N}(R/A) \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, N)$$

Use Proposition 2.3 ■

Proposition 4.5. Let R be excellent equicharacteristic CM with an isolated singularity and k perfect. Then $\mathrm{CM}(R)$ is f.g.

Proof. R excellent, an isolated singularity $\rightsquigarrow$ $\widehat{R}$ an isolated singularity

By Lemma 4.4,

- $\mathrm{V}(\operatorname{jac}\widehat{R}) = \operatorname{Sing}\widehat{R} \subseteq \{\widehat{\mathfrak{m}}\}$ $\quad\therefore$ $\operatorname{jac}\widehat{R}$ is $\widehat{\mathfrak{m}}$-primary or unit
- $\operatorname{jac}\widehat{R} \subseteq \operatorname{ann}\operatorname{Ext}(\mathrm{CM}(\widehat{R}), \operatorname{mod}\widehat{R})$

$\therefore\ \exists\, h > 0$ s.t. $\widehat{\mathfrak{m}}^h \cdot \operatorname{Ext}^{>0}_{\widehat{R}}(\mathrm{CM}(\widehat{R}), \operatorname{mod}\widehat{R}) = 0$

$$\begin{aligned} M, N \in \mathrm{CM}(R) &\rightsquigarrow \widehat{M}, \widehat{N} \in \mathrm{CM}(\widehat{R}) \\ &\rightsquigarrow \widehat{\mathfrak{m}}^h \cdot \operatorname{Ext}^{>0}_{\widehat{R}}(\widehat{M}, \widehat{N}) = 0 \\ &\rightsquigarrow \mathfrak{m}^h \cdot \operatorname{Ext}^{>0}_{R}(M, N) = 0 \\ &\rightsquigarrow \mathfrak{m}^h \subseteq \operatorname{ann}\operatorname{Ext}\mathrm{CM}(R) \end{aligned}$$

Proposition 4.2 $\Rightarrow$ $\mathrm{CM}(R)$ is f.g. ■

Remark 4.6. (1) "perfect" can be removed by using the Cohen–Gabber theorem and the methods of Iyengar–T (2016)
(2) "isolated singularity" can be removed by Dao–T (2014) and Iyengar–T (2016)

Theorem 4.7 (Refinement of Theorem 3.7)**.** Let $(R, \mathfrak{m})$ be a CM local ring. Consider:

(a) $\mathrm{CM}_0(R)$ is finitely generated
(b) $\operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R)$ is $\mathfrak{m}$-primary
(c) $\operatorname{ann}\operatorname{Tor}\mathrm{CM}_0(R)$ is $\mathfrak{m}$-primary
(d) R has an isolated singularity

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). If R is excellent and equicharacteristic, (a)$\sim$(d) are all equivalent.

Proof. Combine Theorem 3.7, Propositions 4.2, 4.5, Remark 4.6(1) ■

Question 4.8. Does (c) $\Rightarrow$ (b)? More strongly, is $\operatorname{ann}\operatorname{Ext}\mathrm{CM}_0(R) = \operatorname{ann}\operatorname{Tor}\mathrm{CM}_0(R)$?

5. Dimension for subcategories of modules

Definition 5.1. For $\mathcal{X} \subseteq \operatorname{mod} R$,

$$\dim \mathcal{X} = \inf\{n \geqslant 0 \mid \exists\, G \in \operatorname{mod} R \text{ s.t. } [G]_{n+1} = \mathcal{X}\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$$

the *dimension* of $\mathcal{X}$ in $\operatorname{mod} R$

Remark 5.2. $\dim \mathcal{X} < \infty \iff \mathcal{X}$ is f.g.

Definition 5.3. Suppose R is Gorenstein and set $(-)^* = \operatorname{Hom}_R(-, R)$

For $M \in \mathrm{CM}(R)$, define the *nth cosyzygy* inductively as follows:

$$\Omega^{-1} M := (\Omega(M^*))^*, \qquad \Omega^{-n} M := \Omega^{-1}(\Omega^{-(n-1)} M) \text{ for } n \geqslant 2$$

Remark 5.4. (1) Let F, G be minimal free resolutions of M, M^*. Then

$$\begin{array}{ccccccccccccccc} \cdots \longrightarrow F_2 & & \longrightarrow & & F_1 & & \longrightarrow & F_0 & \longrightarrow & G_0^* & & \longrightarrow & G_1^* & \longrightarrow & G_2^* \longrightarrow \cdots \\ & \searrow & & \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & \searrow \quad \nearrow & & \\ & & \Omega^2 M & & & \Omega M & & M & & \Omega^{-1} M & & & \Omega^{-2} M & & \end{array}$$

is a minimal free exact complex

(2) M is indecomposable $\Leftrightarrow$ ΩM is indecomposable $\Leftrightarrow$ $\Omega^{-1} M$ is indecomposable

Proposition 5.5. Let R be CM

(1) If R has finite representation type, $\dim \mathrm{CM}(R) = 0$
(2) The converse holds if R is complete and Gorenstein

Proof. (1) Write

$$\operatorname{ind} \mathrm{CM}(R) = \{G_1, \dots, G_n\}$$

$$G := G_1 \oplus \cdots \oplus G_n \ \rightsquigarrow\ \mathrm{CM}(R) = \operatorname{add} G \subseteq [G] \subseteq \mathrm{CM}(R) \ \rightsquigarrow\ \mathrm{CM}(R) = [G] = [G]_1$$

(2) Suppose $\mathrm{CM}(R) = [G]$ for some $G \in \mathrm{CM}(R)$
$G_1, \dots, G_n$ the indecomposable direct summands of G
May assume $G = G_1 \oplus \cdots \oplus G_n$
By Remark 5.4:

$$\operatorname{ind} \mathrm{CM}(R) = \{R\} \cup \{\, \Omega^i G_j \mid i \geqslant 0,\ 1 \leqslant j \leqslant n \,\}.$$

May assume

$$\Omega^i G_j \not\cong G_{j'} \ \forall\, i \geqslant 0, 1 \leqslant \forall\, j \neq \forall\, j' \leqslant n$$

Fix $1 \leqslant j \leqslant n$

$$\Omega^{-1} G_j \cong \Omega^a G_b \ (\exists\, a \geqslant 0,\ 1 \leqslant \exists\, b \leqslant n) \ \rightsquigarrow\ G_j \cong \Omega^{1+a} G_b \ \rightsquigarrow\ b = j$$

$\therefore$ G_j is periodic $\therefore$ $\# \operatorname{ind} \mathrm{CM}(R) < \infty$ ■

Definition 5.6. A CM local ring R has *countable representation type* if $\operatorname{ind} \mathrm{CM}(R)$ is countably infinite

Proposition 5.7. Let R be a complete equicharacteristic hypersurface, $k = \overline{k}$, $\operatorname{char} k \neq 2$. If R has countable representation type, $\dim \mathrm{CM}(R) = 1$

Proof. Araya–Iima–T (2012)

- Buchweitz–Greuel–Schreyer (1987): classify R for all d and $\operatorname{ind} \mathrm{CM}(R)$ for $d = 1$
- Burban–Drozd (2008): classify $\operatorname{ind} \mathrm{CM}(R)$ for $d = 2$
- Reduce to $d = 1, 2$ by Knörrer's periodicity

$\rightsquigarrow$ $\exists\, G$ s.t. $\mathrm{CM}(R) = [G]_2$ ■

Example 5.8. Let $R = \mathbb{C}[[x, y]]/(x^2)$. Then $\mathrm{CM}(R) = [(x)]_2$, $(x) \notin \mathrm{CM}_0(R)$

$$\begin{cases} \dim \mathrm{CM}(R) = 1 \\ \dim \mathrm{CM}_0(R) = \infty \text{ by Theorem 4.7} \end{cases}$$

Definition 5.9 (Artin (1966))**.** A 2-dimensional normal local ring R has a *rational singularity* if $\exists\, X \to \operatorname{Spec} R$ a resolution of singularities s.t. $\mathrm{H}^1(X, \mathcal{O}_X) = 0$

Proposition 5.10. Let R be 2-dimensional complete local normal $\mathbb{C}$-algebra with a rational singularity. Then $\dim \mathrm{CM}(R) \leqslant 1$

Proof. Set

$$\begin{aligned}\Omega\mathrm{CM}(R) &= \mathrm{add}\{R,\, \Omega M \mid M \in \mathrm{CM}(R)\} \\ &= \{X \in \mathrm{mod}\, R \mid \exists\, 0 \to X \to R^{\oplus} \to Y \to 0 \text{ s.t. } Y \in \mathrm{CM}(R)\}\end{aligned}$$

By Iyama–Wemyss (2010):

$$\#\,\mathrm{ind}\,\Omega\mathrm{CM}(R) < \infty$$

Hence $\exists\, G \in \Omega\mathrm{CM}(R)$ s.t. $\Omega\mathrm{CM}(R) = \mathrm{add}\, G$

Fix $M \in \mathrm{CM}(R) \rightsquigarrow M^{\dagger} := \mathrm{Hom}_R(M, \omega) \in \mathrm{CM}(R)$ with ω canonical

$$\exists\, 0 \to N \to R^{\oplus} \to M^{\dagger} \to 0,$$

where $N := \Omega(M^{\dagger}) \in \Omega\mathrm{CM}(R) = \mathrm{add}\, G$

$$\exists\, 0 \to M \to \omega^{\oplus} \to L \to 0,$$

where $L := N^{\dagger} \in \mathrm{add}(G^{\dagger})$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & \omega^{\oplus} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & R^{\oplus} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Omega L & = & \Omega L & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\quad \rightsquigarrow \quad \begin{cases} K \cong M \oplus R^{\oplus} \\ \exists\, 0 \to \Omega L \to M \oplus R^{\oplus} \to \omega^{\oplus} \to 0 \end{cases}
$$

$\Omega L \in \mathrm{add}(\Omega(G^{\dagger})) \subseteq [G^{\dagger}]$ $\quad \therefore\ M \in [G^{\dagger} \oplus \omega]_2 \quad \therefore\ \mathrm{CM}(R) = [G^{\dagger} \oplus \omega]_2$ ■

Theorem 5.11.

(1) Let R be an excellent equicharacteristic local hypersurface with an isolated singularity. Then

$$\dim \mathrm{CM}(R) \leqslant \begin{cases} 2\,\ell\ell(R/J) - 1 & \\ \mathrm{e}(J) - 1 & (\text{if } |k| = \infty) \end{cases}$$

where $J = \mathrm{jac}\, R$

(2) Let R be a complete intersection. Then

$$\dim \mathrm{CM}(R) \geqslant \mathrm{codim}\, R - 1$$

Proof. The stable category $\underline{\mathrm{CM}}(R)$ is triangulated as R is Gorenstein

Consider the dimension of $\underline{\mathrm{CM}}(R)$ in the sense of Rouquier (2008)

By Dao–T (2015[1])

- $\dim \mathrm{CM}(R) \geqslant \dim \underline{\mathrm{CM}}(R)$ whenever R is Gorenstein

- The equality holds if R is a hypersurface

(1) Apply Dao–T (2015[2])
(2) Apply Avramov–Iyengar (2006∼) ■

Problems.

(1) Give upper/lower bounds for $\dim \mathrm{CM}(R)$ in other cases. Start by giving:
- a lower bound when R is Gorenstein, non-CI
- an upper bound when R is a $\begin{cases}\text{hypersurface not having an isolated singularity}\\ \text{non-hypersurface CI having an isolated singularity}\end{cases}$
- an example of CM local rings R with $\dim \mathrm{CM}(R) = 2$

(2) $M \in \operatorname{mod} R$ is *totally reflexive* if $\begin{cases} M \cong M^{**} \\ \operatorname{Ext}^{>0}(M, R) = 0 \\ \operatorname{Ext}^{>0}(M^*, R) = 0 \end{cases}$

Let
$$\mathcal{G}(R) = \{\text{Totally reflexive } R\text{-modules}\}$$
We have:
- $\mathcal{G}(R) \subseteq \mathrm{CM}(R)$ if R is CM
- The equality holds if R is Gorenstein

$$\begin{cases} \mathcal{G}(R) \neq \operatorname{add} R \\ \dim \mathcal{G}(R) < \infty \end{cases} \implies \text{is } R \text{ Gorenstein?}$$

(a) Christensen–Piepmeyer–Striuli–T (2008):
$$\begin{cases} \mathcal{G}(R) \neq \operatorname{add} R \\ \# \operatorname{ind} \mathcal{G}(R) < \infty \end{cases} \implies R \text{ is Gorenstein}$$

(b) If $\mathcal{G}(R) = \operatorname{add} R$, $\dim \mathcal{G}(R) = 0 < \infty$ but R is not necessarily Gorenstein (e.g. any non-Gorenstein R with $\mathfrak{m}^2 = 0$ satisfies this)

Exercises

In what follows, let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. All modules are assumed to be finitely generated.

1. Let M be an R-module.

(1) Prove that for any proper ideal I of R one has

$$\dim(0 :_M I) \leqslant \dim M/IM.$$

(2) If $x \in R$ is a subsystem of parameters of M, then show that

$$\dim(0 :_M x) < \dim M.$$

2. Let k be a field.

(1) Let $R = k[[x, y]]/(xy)$. Show that

$$\operatorname{ann} \operatorname{Tor} \operatorname{CM}_0(R) = \operatorname{ann} \operatorname{Ext} \operatorname{CM}_0(R) = (x, y)$$

by using the fact that $\operatorname{ind} \operatorname{CM}(R) = \{R, R/(x), R/(y)\}$.

(2) Let $R = k[[x, y]]/(x^2)$. Show that

$$\operatorname{ann} \operatorname{Tor} \operatorname{CM}_0(R) = \operatorname{ann} \operatorname{Ext} \operatorname{CM}_0(R) = (x)$$

by using the fact that $\operatorname{ind} \operatorname{CM}(R) = \{R, (x), (x, y), (x, y^2), (x, y^3), \dots\}$.

3. Let M be an R-module. Show the following.

(1) Let

$$\cdots \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be an exact sequence of R-modules such that P_i is free for all $i \geqslant 0$. Then for each $n \geqslant 0$ there exists $t \geqslant 0$ such that $\operatorname{Im} f_n \cong \Omega^n M \oplus R^{\oplus t}$.

(2) Let $n \geqslant 0$ be an integer. For each $\mathfrak{p} \in \operatorname{Spec} R$ there is an integer $t \geqslant 0$ such that

$$(\Omega_R^n M)_{\mathfrak{p}} \cong \Omega_{R_{\mathfrak{p}}}^n (M_{\mathfrak{p}}) \oplus R_{\mathfrak{p}}^{\oplus t}.$$

(3) Suppose that R is Cohen–Macaulay. Let $n \geqslant d$ be an integer.
   (a) The module $\Omega^n M$ is in $\operatorname{CM}(R)$.
   (b) Let M be an R-module of finite length. Then $\Omega^n M$ is in $\operatorname{CM}_0(R)$.

4. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Let $n \geqslant 0$ be an integer. Then show that applying $\Omega^n(-)$ to this induces an exact sequence of the form

$$0 \to \Omega^n L \to \Omega^n M \oplus R^{\oplus t} \to \Omega^n N \to 0.$$

5. Verify that there are isomorphisms

$$\operatorname{Tor}_n^R(\Omega^t M, N) \cong \operatorname{Tor}_{n+t}^R(M, N), \qquad \operatorname{Ext}_R^n(\Omega^t M, N) \cong \operatorname{Ext}_R^{n+t}(M, N)$$

for all $n, t \in \mathbb{Z}_{>0}$ and $M, N \in \operatorname{mod} R$.

6. Let $L \to M \to N$ be an exact sequence of R-modules. Then show:

$$\operatorname{ann}_R L \cdot \operatorname{ann}_R N \subseteq \operatorname{ann}_R M.$$

7. Let M, N be R-modules. Show that $\operatorname{P}_R(M, N)$ is an R-submodule of $\operatorname{Hom}_R(M, N)$.

8. Prove Lemma 2.2.

9. Prove Lemma 2.5.

10. Prove Remark 3.2.

11. Let k be a field, and let n be a nonnegative integer. Let $R = k[x]/(x^{n+1})$ be a quotient of a polynomial ring. Prove the following.

(1) Let $a = a_0 + a_1x + \cdots + a_nx^n$ be an element of R. Then a is a unit of R if and only if $a_0 \neq 0$.

(2) An R-module is indecomposable if and only if it is isomorphic to $R/(x^i)$ for some $1 \leqslant i \leqslant n+1$.

(3) For each integer $1 \leqslant i \leqslant n$ there exists an exact sequence

$$0 \to R/(x^i) \to R/(x^{i+1}) \oplus R/(x^{i-1}) \to R/(x^i) \to 0,$$

where we set $x^0 := 1$.

(4) For every $1 \leqslant i \leqslant n$ there exists $j \geqslant 1$ such that $[R/(x^i)]_j = \operatorname{mod} R$.

12. Suppose that R is Cohen–Macaulay. Show the following.

(1) Let $M, N \in \operatorname{mod} R$. Then $M, N \in \mathrm{CM}_0(R)$ if and only if $M \oplus N \in \mathrm{CM}_0(R)$.

(2) Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules with $N \in \mathrm{CM}_0(R)$. Then $L \in \mathrm{CM}_0(R)$ if and only if $M \in \mathrm{CM}_0(R)$.

(3) For all $M \in \mathrm{CM}_0(R)$ one has $\Omega M \in \mathrm{CM}_0(R)$.

13. Suppose that R is Cohen–Macaulay. Show that R has an isolated singularity if and only if $\mathrm{CM}_0(R) = \mathrm{CM}(R)$.

14. Let R be a Gorenstein local ring. Show the following.

(1) Let $M, N \in \mathrm{CM}_0(R)$. For all integers $i > 0$ there are isomorphisms

$$\operatorname{Ext}_R^d(\operatorname{Tor}_i^R(M, N^*), R) \cong \operatorname{Ext}_R^{d+i}(M, N),$$
$$\operatorname{Ext}_R^d(\operatorname{Ext}_R^{d+i}(M, N^*), R) \cong \operatorname{Tor}_i^R(M, N).$$

(2) There is an equality

$$\operatorname{ann} \operatorname{Tor} \mathrm{CM}_0(R) = \operatorname{ann} \operatorname{Ext} \mathrm{CM}_0(R).$$

15. Let $S = k[[x, y]]$ be a formal power series ring over a field k, and let $R = k[[x^3, x^2y, xy^2, y^3]]$ be the third Veronese subring of S. (To be precise, R is the completion of the third Veronese subring of $k[x, y]$.) Prove the following.

(1) There is a direct sum decomposition $S = R \oplus W \oplus P$, where $W = Rx + Ry$ and $P = Rx^2 + Rxy + Ry^2$ are the R-submodules of S.

(2) For each $M \in \mathrm{CM}(R)$, the R-module $\operatorname{Hom}_R(S, M)$ is a direct sum of copies of S.

(3) One has $\operatorname{ind} \mathrm{CM}(R) = \{R, W, P\}$ and $\dim \mathrm{CM}(R) = 0$.

(4) There are R-isomorphisms $\Omega W \cong P$ and $\Omega P \cong P^{\oplus 2}$.

(5) One has $\operatorname{ind} \Omega\mathrm{CM}(R) = \{R, P\}$.

16. Let $R = \mathbb{C}[[x, y]]/(x^3 + y^3)$ be a quotient of a formal power series ring. Set $J = \operatorname{jac} R$.

(1) Find $\mathrm{e}(J)$ and $\ell\ell(R/J)$.

(2) Evaluate $\dim \mathrm{CM}(R)$.

References

[1] T. Araya; K.-i. Iima; R. Takahashi, On the structure of Cohen–Macaulay modules over hypersurfaces of countable Cohen–Macaulay representation type, *J. Algebra* **361** (2012), 213–224.
[2] M. Artin, On isolated rational singularities of surfaces, *Amer. J. Math.* **88** (1966), 129–136.
[3] M. Auslander, Isolated singularities and existence of almost split sequences, *Representation theory, II (Ottawa, Ont., 1984)*, 194–242, Lecture Notes in Math., 1178, *Springer, Berlin*, 1986.
[4] P. A. Bergh; S. B. Iyengar; H. Krause; S. Oppermann, Dimensions of triangulated categories via Koszul objects, *Math. Z.* **265** (2010), no. 4, 849–864.
[5] R.-O. Buchweitz; G.-M. Greuel; F.-O. Schreyer, Cohen–Macaulay modules on hypersurface singularities, II, *Invent. Math.* **88** (1987), no. 1, 165–182.
[6] I. Burban; Y. Drozd, Maximal Cohen–Macaulay modules over surface singularities, *Trends in representation theory of algebras and related topics*, 101–166, EMS Ser. Congr. Rep., *Eur. Math. Soc., Zürich*, 2008.
[7] L. W. Christensen; G. Piepmeyer; J. Striuli; R. Takahashi, Finite Gorenstein representation type implies simple singularity, *Adv. Math.* **218** (2008), no. 4, 1012–1026.
[8] H. Dao; R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory* **8** (2014), no. 1, 141–172.
[9] H. Dao; R. Takahashi, The dimension of a subcategory of modules, *Forum Math. Sigma* **3** (2015), e19, 31 pp.
[10] H. Dao; R. Takahashi, Upper bounds for dimensions of singularity categories, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 4, 297–301.
[11] C. Huneke; G. J. Leuschke, Two theorems about maximal Cohen-Macaulay modules, *Math. Ann.* **324** (2002), no. 2, 391–404.
[12] O. Iyama; M. Wemyss, The classification of special Cohen–Macaulay modules, *Math. Z.* **265** (2010), no. 1, 41–83.
[13] S. B. Iyengar; R. Takahashi, Annihilation of cohomology and strong generation of module categories, *Int. Math. Res. Not. IMRN* **2016**, no. 2, 499–535.
[14] H. Knörrer, Cohen–Macaulay modules on hypersurface singularities, I, *Invent. Math.* **88** (1987), no. 1, 153–164.
[15] G. Leuschke; R. Wiegand, Ascent of finite Cohen-Macaulay type, *J. Algebra* **228** (2000), no. 2, 674–681.
[16] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, 8, *Cambridge University Press, Cambridge*, 1989.
[17] R. Rouquier, Dimensions of triangulated categories, *J. K-Theory* **1** (2008), no. 2, 193–256.
[18] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, *Pacific J. Math.* **268** (2014), no. 1, 231–248.
[19] H.-J. Wang, On the Fitting ideals in free resolutions, *Michigan Math. J.* **41** (1994), no. 3, 587–608.
[20] Y. Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, London Mathematical Society Lecture Note Series, 146, *Cambridge University Press, Cambridge*, 1990.

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

E-mail address: `takahashi@math.nagoya-u.ac.jp`

URL: `http://www.math.nagoya-u.ac.jp/~takahashi/`